Variational Fluid Mechanics

Harry Winston Sullivan

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1 Introduction

A primary goal in the study of fluids is to describe the motion of a fluid system in response forces. There are two main paradigms in it's parameterization, the spacial Eulerian description and the material Lagrangian description. Despite their differences, it is clear that they must agree in their predictions. This would imply there is some method of transforming back and forth between them. Despite the plausibility of such a transformation, simple ideas in one description turn out to be not so simple in the other. A primary example of this is the principle of least action and Hamilton's equations of motion from classical physics with point particles. To extend the particle based formalism into a Lagrangian description one just needs replace the summation with an integral over the material body. However, if you apply this formalism directly to the Eulerian description the solutions are always the trivial motionless velocity field $v_i = 0$. How does the change of parameterization cause this to occur? Similarly, what can we do to fix the Eulerian perspective to reproduce the same predictions as the Lagrangian one?

1.1 How to Describe a Fluid

The Lagrangian description is commonly on the intuitive side for those who come from a classical physics background. Similar to a classical physics system, we have an origin O and there is some identifier $\mathbf{X}$ which indicates a particular constituent of our system. However, rather than the constituent being a point particle, it is a element in a continuum. The common way to define $\mathbf{X}$ as some vector that points from O to the element at some point P_0 and some reference time $t = 0$. Using this description we can then consider a function $\mathbf{x}$ that will indicate the current spacial position P of the original element $\mathbf{X}$ at some later time t.

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{1}$$

These are the two types of co-ordinates used in fluid mechanics, the spacial (Eulerian) co-ordinates $\mathbf{x}$ and the material (Lagrangian) co-ordinates $\mathbf{X}$. In principle it is possible to flip equation 1 around. If we knew the path of every

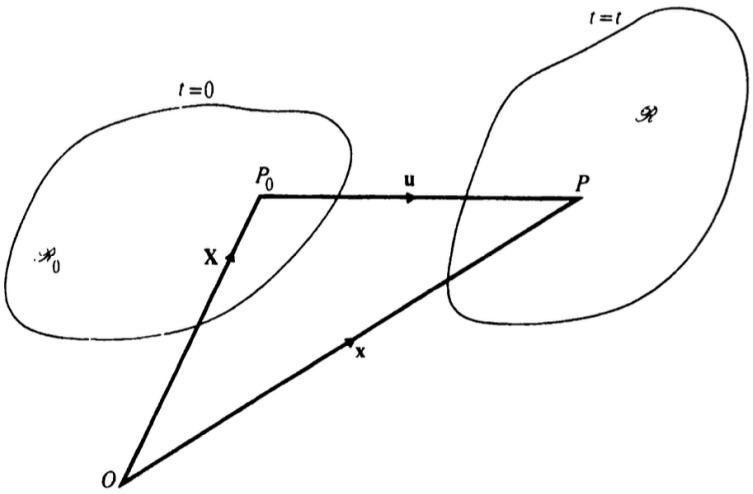


Figure 1: A reference configuration and it's corresponding configuration at time t.

fluid element, we could indicate what fluid element was at a position $\mathbf{x}$ and some time t.

$$\mathbf{X} = \mathbf{X}(\mathbf{x}, t) \tag{2}$$

We can now use these to describe the displacement $\mathbf{u}$.

$$\mathbf{u} = \mathbf{x} - \mathbf{X} \tag{3}$$

The displacement can be thought of from either perspective, all it amounts to is deciding what is a function of what. Ex:

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{x} - \mathbf{X}(\mathbf{x}, t) \tag{4}$$
$$\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X} \tag{5}$$

With this type of definition on what is a function of what, somewhat awkwardly, we can also write:

$$\mathbf{x} = \mathbf{x}(\mathbf{X}(\mathbf{x}, t), t) \tag{6}$$
$$\mathbf{X} = \mathbf{X}(\mathbf{x}(\mathbf{X}, t), t) \tag{7}$$

This allows us to discuss the distortion terms. The distortion can be used to quantify the transformation between a Lagrangian and an Eulerian frame of

reference. The main quantity used is the Jacobian[1], defined as follows:

$$\det(\hat{\mathbf{J}}) = \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix} \tag{8}$$

It is defined in terms of a matrix of derivatives. The vertical bars in the matrix[2] indicate a determinant. There is also the inverse of this matrix:

$$\det(\hat{\mathbf{J}}^{-1}) = \frac{\partial(X_1, X_2, X_3)}{\partial(x_1, x_2, x_3)} \tag{9}$$

Using this matrix, we can define a few different transformation rules that will benefit us later. The first of which is obvious by definition:

$$\hat{\mathbf{J}}^{-1}\hat{\mathbf{J}} = \hat{\mathbf{J}}\hat{\mathbf{J}}^{-1} = \mathbf{1} \tag{10}$$

Also note this relation:

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}) \tag{11}$$

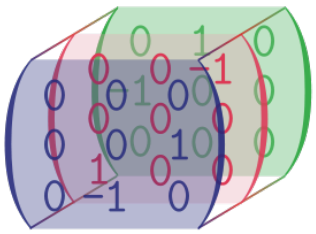
$$\implies \det(\hat{\mathbf{J}})\det(\hat{\mathbf{J}}^{-1}) = \det(\mathbf{1}) = 1 \tag{12}$$

We can define the Levi-Civita tensor and symbol to aid in our analysis of the Jacobian:

$$\epsilon_{i,j,k} = \left\{ \begin{array}{ll} 1, & \text{if } i,j,k \text{ is an even permutation} \\ -1, & \text{if } i,j,k \text{ is an odd permutation} \\ 0, & \text{else} \end{array} \right\} \tag{13}$$

Where an even permutation is any order of i, j, k that makes something that resembles $(1, 2, 3)$ in a cyclic way $(2, 3, 1), (3, 1, 2)$. Similarly the odd ones are when you have the opposite $(3, 2, 1), (2, 1, 3)$, and $(1, 3, 2)$. If any indices are repeated you get 0. This defines a tensor[3] as well, in that you can put the corresponding in a three dimensional "matrix" $\hat{\boldsymbol{\epsilon}_{i,j,k}}$[4]:

$$\epsilon_{ijk} =$$



[1]The Jacobian is also used when transforming PDFS of random varaibles. If you have a function of a random variable and you want to see the distribution of the transformed one, the determinant of the Jacobian will be used to conserve probability throughout the transformation. In that it ensures the integral of the distribution is still 1 after transforming.

[2]Some may define the Jacobian matrix as the transpose of the matrix above, beware.

[3]There is a better definition of what a tensor is. There is special transformation and index rules. In all honesty it makes linear algebra much easier to understand from this perspective. For some notes on that see the great videos by eigenchris: https://www.youtube.com/playlist?list=PLJHszsWbB6hrkmmq57lX8BV-o-YIOFsiG

[4]The formatting here looks jank asf but I don't care to fix it.

Then we can also remember that a derivative of a vector input, scalar output function is a vector. Similarly, if we treat the the determinant of the Jacobian as a function of a matrix, we obtain a matrix when we differentiate. This shape property can spark an idea, is it possible to write the inverse of a matrix in terms of derivatives of the determinant? To cheat and look in the back of the book, it has been shown that (where $a_{i,j}$ is a component of $\mathbf{A}$):

$$\hat{\mathbf{A}}^{-1} = \frac{1}{\det(\hat{\mathbf{A}})} \frac{\partial \det(\hat{\mathbf{A}})}{\partial a_{i,j}} \tag{14}$$

To show this, first verify the Levi-Civita symbols use in defining the determinant:

$$\det(\hat{\mathbf{A}}) = \epsilon_{i,j,k} a_{1,i} a_{2,j} a_{3,k} \tag{15}$$

Noting first that if I repeat variables then I am implicitly summing over them. Then we can differentiate this expression with respect to $a_{1,i}$:

$$\frac{\partial \det(\hat{\mathbf{A}})}{\partial a_{1,j}} = \epsilon_{i,j,k} a_{2,j} a_{3,k} \tag{16}$$

We can then multiply by $\frac{\det(\hat{\mathbf{A}})}{a_{1,j}}$ to see that:

$$\frac{1}{\det \hat{\mathbf{A}}} \frac{\partial \det(\hat{\mathbf{A}})}{\partial a_{1,j}} a_{1,j} = 1 \tag{17}$$

Similarly, it is obvious that we can exchange the index 1 for some k:

$$\frac{1}{\det \hat{\mathbf{A}}} \frac{\partial \det(\hat{\mathbf{A}})}{\partial a_{k,j}} a_{k,j} = 1 \tag{18}$$

Then we can see if the derivative index doesn't match the index of the row component we get:

$$\frac{1}{\det \hat{\mathbf{A}}} \frac{\partial \det(\hat{\mathbf{A}})}{\partial a_{k,j}} a_{l,j} \text{ such that } k \neq l \tag{19}$$

$$= 0 \tag{20}$$

Showing a specific example by taking equation 16 and multiplying by a_{2_i} and dividing by the determinant.

$$\frac{\partial \det(\hat{\mathbf{A}})}{\partial a_{1,j}} \frac{a_{2,i}}{\det(\hat{\mathbf{A}})} = \frac{\epsilon_{i,j,k} a_{2,i} a_{2,j} a_{3,k}}{\det(\hat{\mathbf{A}})} = \frac{\epsilon_{i,j,k} a_{2,i} a_{2,j}}{\epsilon_{i,j,k} a_{1,i} a_{2,j}} \tag{21}$$

Then expanding the numerator on its own will give:

$$\frac{\partial \det(\hat{\mathbf{A}})}{\partial a_{1,j}} \frac{a_{2,i}}{\det(\hat{\mathbf{A}})} = 0 \tag{22}$$

This is shown similarly for other indices. This then implies that a row of the the inverse is:

$$\frac{1}{\det \hat{\mathbf{A}}} \frac{\partial \det(\hat{\mathbf{A}})}{\partial a_{1,i}} = a_{1,i}^{-1} \tag{23}$$

Convince yourself by remembering what a matrix does when multiplied by its inverse. This now allows us to exploit this to convert a back and forth between representations with the Jacobian matrix determinant. If we differentiate the Jacobian matrix with respect to the component we will find:

$$\hat{\mathbf{J}}^{-1} = \frac{1}{\det(\hat{\mathbf{J}})} \frac{\partial \det(\hat{\mathbf{J}})}{\partial(\frac{\partial x_i}{\partial X_j})} \tag{24}$$

This relationship will prove useful later on.

We can denote the velocity of a fixed material element at some time t as the time derivative of the displacement.

$$\mathbf{v}(\mathbf{X}, t) = \frac{\partial \mathbf{u}(\mathbf{X}, t)}{\partial t} = \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t} \tag{25}$$

However, we may want to instead see what the velocity is at a fixed point in space $\mathbf{x}$. To achieve this we have to introduce the material derivative. We can regard a non-specific quantity ϕ as some observable in our body that will vary throughout space and time. ϕ is not necessarily in either description, we can evaluate it in either perspective:

$$\phi = G(\mathbf{X}, t) = g(\mathbf{x}, t) \tag{26}$$

Suppose we wish to see how ϕ changes in time. It is clear from the velocity example that:

$$\dot{\phi} = \frac{\partial G(\mathbf{X}, t)}{\partial t} = \frac{\partial G(\mathbf{X}(\mathbf{x}, t), t)}{\partial t} \tag{27}$$

This IS the material derivative [5], a simple time rate of change of a quantity parameterized by material elements. The common notation for this is:

$$\frac{D\phi}{Dt} = \dot{\phi} = \frac{\partial G(\mathbf{X}, t)}{\partial t} \tag{28}$$

By considering the spacial description $g(x, t)$ and the chain rule[6] gives us:

$$\frac{D\phi}{Dt} = \frac{g(\mathbf{x}, t)}{\partial t} + \sum_{i=1}^{3} \frac{\partial g(\mathbf{x}, t)}{\partial x_i} \frac{\partial x_i(\mathbf{X}, t)}{\partial t} \tag{29}$$

[5] Also known as the convected or the barycentric derivative.
[6] i.e. $g(\mathbf{x}, t) = g(x(\mathbf{X}, t), t)$

Interestingly enough, the quantity $\frac{\partial x_i(\mathbf{X},t)}{\partial t}$ is the velocity component v_i. This allows us to write:

$$\frac{D\phi}{Dt} = \mathbf{v} \cdot \nabla g(\mathbf{x}, t) + \frac{g(\mathbf{x}, t)}{\partial t} \tag{30}$$

Often, people take conservation laws (energy, mass, momentum) in the Lagrangian description and then convert them to a spacial description with the material derivative. Overall abandoning any discussion of material co-ordinates from then on. However, it becomes important to consider the Lagrangian description in more depth when translating a functional.

2 Functional Derivatives and Minimization

In classical mechanics, the quantity $L = T - U$ is named the Lagrangian. It is a function of the generalized path of a particle $\mathbf{q}(t)$, it's derivative $\dot{\mathbf{q}}(t)$, and time explicitly. By integrating the Lagrangian in time we can get the action of a system.

$$S[\mathbf{q}] = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t)dt \tag{31}$$

The action is a number, and the Lagrangian is a function, if the action is a function of the Lagrangian then it must be a *functional*. Another example of a functional could be something as simple as the 2d arc-length s of a planar curve $y = f(x)$[7].

$$s[f(x)] = \int_{x_0}^{x_1} \sqrt{1 + \left|\frac{df(x)}{dx}\right|^2}\, dx \tag{32}$$

Another possible example could be the error between some model some data points.

$$\chi^2 = \sum_{i=1}^{N} \frac{(f(x_i) - y_i)^2}{2\sigma} \tag{33}$$

Often when considering an error functional, we aim to find a function $f(x)$ that will cause the functional to be at a minimum. Through the minimization of the arc-length we can arrive at a method for determining the path light travels. In fact, most physical laws can be written in terms of the minimization of some functional. In the case of the action functional we will see that it gives the equations of motion. The question now is, how can we actually minimize a functional? and can we adapt the action functional into an Eulerian perspective?

[7]The length of an infinitesimal segment of the curve is given by $\sqrt{dx^2 + dy^2}$. We can multiply by $\frac{dx}{dx}$, $\sqrt{dx^2 + dy^2} = \sqrt{1 + \left|\frac{dy}{dx}\right|^2}\, dx$, and "sum" them to get the arc-length s.

To minimize a regular function we can use a derivative. A functional can have a derivative in a similar fashion. Rather than varying a scalar input, we can vary an entire function like what is seen in figure 2.

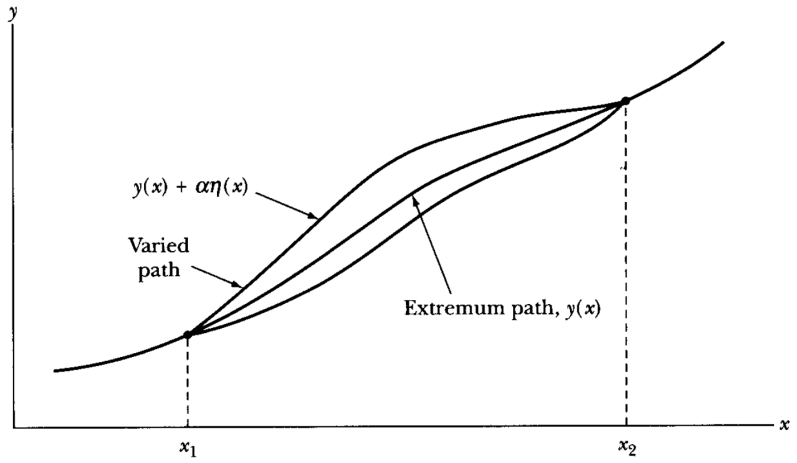


Figure 2: $y(x)$ is a function that will minimize some functional. The function $y(x) + \alpha\eta(x)$ is some small variation that equals $y(x)$ at the end points.

To define the functional derivative[8] we can take some ideas from functional analysis, the basic definition for the variation in a function is:

$$\delta F[g(x)] = \lim_{\epsilon \to 0} \frac{F[g(x) + \epsilon\eta(x)] - F[g(x)]}{\epsilon} \tag{34}$$

This looks awfully similar to the usual definition of a derivative of a regular function:

$$\frac{df(x)}{dx} = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon} \tag{35}$$

The variation is actually just the derivative of the functional with respect to some epsilon evaluated at zero.

$$\delta F[g(x)] = \left. \frac{dF[g(x) + \epsilon\eta(x)]}{d\epsilon} \right|_{\epsilon=0} \tag{36}$$

But this is analogous to the total derivative, in that the total derivative of a regular function F is expanded like $dF = \sum_{i=0}^{N} \frac{dF}{dx_i} dx_i$. The function $\eta(x)dx$ is the variation dx_i in function space. If we had a total derivative, we know that the following:

$$\delta F[g(x)] = \int \frac{\delta F[g(x)]}{\delta g(x)} \eta(x) dx \tag{37}$$

[8]Sometimes the functional derivative can be considered a Gateaux derivative or a Fréchet derivative.

If this is unsatisfying to you, I agree. It's just a definition, take it as truth and move on with your life. As an engineer we should be concerned with the applicability of the method for predictions. As an example we could consider the functional $F[g(x)]$ to be defined as:

$$F[g(x)] = \int_0^1 (g(x))^3 dx \tag{38}$$

Then we can perturb the function g by some function $\eta(x)$:

$$F[g(x) + \epsilon\eta(x)] = \int_0^1 (g(x) + \eta(x))^3 dx \tag{39}$$

Using ordinary calculus, the derivative of this functional with respect to the scalar ϵ is:

$$\frac{dF[g(x) + \epsilon\eta(x)]}{d\epsilon} = \int_0^1 3\rho(x)(g(x) + \epsilon g(x))^2 dx \tag{40}$$

Taking the limit:

$$\lim_{\epsilon\to 0} \int_0^1 3\eta(x)(g(x))^2 dx \tag{41}$$

But this is equal to the integral in equation 37. Along with this it must hold for any function η.

$$\lim_{\epsilon\to 0} \int_0^1 3\eta(x)(g(x))^2 dx = \int \frac{\delta F[g(x)]}{\delta g(x)} \eta(x) dx \tag{42}$$

$$\implies \frac{\delta F[g(x)]}{\delta g(x)} = 3(g(x))^2 \tag{43}$$

An interesting note, this is a function! The derivative of a functional is a function. We gain some dimensionality when we take a the derivative. This is similar to taking the gradient of a scalar output vector input function. However it is often helpful to consider just the variation on its own. to acquire an easier formula to perform this type of optimization. If we consider the variation as zero we can see the following:

$$\delta F[g(x)] = \frac{d}{d\epsilon} \int_{x_1}^{x_2} f(g(x), g'(x); x) dx \epsilon = 0 \tag{44}$$

Where the little f is understood as some operation that acts on g, such as differentiation. As integral bounds don't change we can move the derivative into the integral.

$$= \int_{x_1}^{x_2} \frac{d}{d\epsilon} f(g(x), g'(x); x) dx \Big|_{\epsilon=0} \tag{45}$$

$$= \int_{x_1}^{x_2} \frac{df}{dg}\frac{dg}{d\epsilon} + \frac{df}{dg'}\frac{dg'}{d\epsilon} dx \Big|_{\epsilon=0} \tag{46}$$

But we know that the variation in the function and its derivative with respect to epsilon is given by:

$$\frac{dg}{d\epsilon} = \eta(x) \tag{47}$$

$$\frac{dg'}{d\epsilon} = \frac{d\eta(x)}{dx} \tag{48}$$

This will remove the epsilon dependence. Putting these in the expression and then integrating by parts gives:

$$= \int_{x_1}^{x_2} \frac{df}{dg}\eta(x) + \frac{df}{dg'}\frac{d\eta(x)}{dx}dx \tag{49}$$

$$= \int_{x_1}^{x_2} \frac{df}{dg}\eta(x) - \frac{d}{dx}\left(\frac{df}{dg'}\right)\eta(x)dx + \frac{df}{dg'}\eta(x)\Big|_{x_1}^{x_2} \tag{50}$$

$$= \int_{x_1}^{x_2} \left(\frac{df}{dg} - \frac{d}{dx}\left(\frac{df}{dg'}\right)\right)\eta(x)dx \tag{51}$$

But this must hold for any choice of variation η. If we assert that this variation must be zero, then we get:

$$\frac{df}{dg} - \frac{d}{dx}\left(\frac{df}{dg'}\right) = 0 \tag{52}$$

This is the Euler equation, if we then take it and apply it to a Lagrangian we get the Euler-Lagrange equation.

As a final example we can take the variation of the action S to be zero and see what the consequence is. Expanding the definition of $T - U$ gives the functional as:

$$S[\mathbf{q}] = \int_{t_1}^{t_2} \frac{1}{2}m\left(\frac{d\mathbf{q}(t)}{dt}\right)^2 - U(\mathbf{q}(t))dt \tag{53}$$

Then the variation in the functional is given as:

$$\delta \int_{t_1}^{t_2} \frac{1}{2}m\left(\frac{d\mathbf{q}(t)}{dt}\right)^2 - U(\mathbf{q}(t))dt \tag{54}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{t_1}^{t_2} \frac{1}{2}m\left(2\frac{d\mathbf{q}(t)}{dt}\epsilon\frac{d\boldsymbol{\eta}(t)}{dt}\right) + \frac{1}{2}m\left(\epsilon\frac{d\boldsymbol{\eta}(t)}{dt}\right)^2 + \frac{1}{2}m\left(\frac{d\mathbf{q}(t)}{dt}\right)^2 \tag{55}$$

$$- U(\mathbf{q}(t) + \epsilon\boldsymbol{\eta}(t)) - \frac{1}{2}m\left(\frac{d\mathbf{q}(t)}{dt}\right)^2 + U(\mathbf{q}(t))dt \tag{56}$$

$$= \lim_{\epsilon\to 0}\int_{t_1}^{t_2} -U(\mathbf{q}(t) + \epsilon\boldsymbol{\eta}(t)) + U(\mathbf{q}(t)) + m\left(\frac{d\mathbf{q}(t)}{dt}\frac{d\boldsymbol{\eta}(t)}{dt}\right) + \mathcal{O}(\epsilon)dt \tag{57}$$

$$= \int_{t_1}^{t_2} -U(\mathbf{q}(t) + \epsilon\boldsymbol{\eta}(t)) + U(\mathbf{q}(t)) + m\left(\frac{d\mathbf{q}(t)}{dt}\frac{d\boldsymbol{\eta}(t)}{dt}\right)dt \tag{58}$$

Then we can separate the integral into the kinetic and potential parts then perform integration by parts to remove the dependence on the derivative of $\boldsymbol{\eta}(t)$.

$$= \int_{t_1}^{t_2} -U(\mathbf{q}(t) + \epsilon\boldsymbol{\eta}(t)) + U(\mathbf{q}(t))dt + m\int_{t_1}^{t_2} \left(\frac{d\mathbf{q}(t)}{dt}\frac{d\boldsymbol{\eta}(t)}{dt}\right)dt \tag{59}$$

$$= \int_{t_1}^{t_2} -U(\mathbf{q}(t) + \epsilon\boldsymbol{\eta}(t)) + U(\mathbf{q}(t))dt - m\int_{t_1}^{t_2} \left(\frac{d^2\mathbf{q}(t)}{dt^2}\boldsymbol{\eta}(t)\right)dt + m\frac{d\mathbf{q}(t)}{dt}\boldsymbol{\eta}(t)\Big|_{t_1}^{t_2} \tag{60}$$

If we restrict ourselves to the space of functions such that endpoints cause the $\boldsymbol{\eta}$ term to vanish, we can just remove it from the analysis.

$$= \int_{t_1}^{t_2} -(U(\mathbf{q}(t) + \epsilon\boldsymbol{\eta}(t)) - U(\mathbf{q}(t))) - m\left(\frac{d^2\mathbf{q}(t)}{dt^2}\boldsymbol{\eta}(t)\right)dt \tag{61}$$

Using equation 37 we get:

$$= \int_{t_1}^{t_2} -(U(\mathbf{q}(t) + \epsilon\boldsymbol{\eta}(t)) - U(\mathbf{q}(t))) - m\left(\frac{d^2\mathbf{q}(t)}{dt^2}\boldsymbol{\eta}(t)\right)dt = \int_{t_1}^{t_2} \frac{\delta S[\mathbf{q}(t)]}{\delta\mathbf{q}(t)}\boldsymbol{\eta}(t)dt \tag{62}$$

This allows us to identify the functional derivative as:

$$\frac{\delta S[\mathbf{q}(t)]}{\delta\mathbf{q}(t)} = -\frac{dU(\mathbf{q}(t))}{d\mathbf{q}(t)} - m\frac{d^2\mathbf{q}(t)}{dt^2} = 0 \tag{63}$$

But the gradient of potential is just the force! Implying:

$$\mathbf{F} = m\frac{d^2\mathbf{q}(t)}{dt^2} \tag{64}$$

This will recover Newtonian mechanics completely. With this, we can move on to the application in a fluid.

3 Variational Calculus for Fluids

3.1 Lagrangian Form

The equations of motion, $F = ma$, for a material element $\mathbf{X}$ in an inviscid flow are written as:

$$\rho\frac{\partial^2\mathbf{x}(\mathbf{X},t)}{\partial t^2} = -\frac{\partial p}{\partial\mathbf{x}(\mathbf{X},t)} \tag{65}$$

In an index format:

$$\rho\frac{\partial^2 x_i(\mathbf{X},t)}{\partial t^2} = -\frac{\partial p}{\partial x_i(\mathbf{X},t)} \qquad (i = 1,2,3) \tag{66}$$

This form of the equations of motion are hard to deal with as the derivative of the pressure depends on the positional co-ordinates $\mathbf{x}$. We can use a Jacobian to relate the change in space to a change in positional co-ordinates. This can be used to quantify how the density changes.

$$\rho = \frac{\partial(X_1, X_2, X_3)}{\partial(x_1, x_2, x_3)}\rho_0(\mathbf{X}) = |\hat{\mathbf{J}}^{-1}|\rho_0(\mathbf{X}) \tag{67}$$

Similarly we can re write the pressure derivative by expanding it into partials:

$$\frac{\partial p}{\partial x_i} = \sum_{j=1}^{3} \frac{\partial X_j}{\partial x_i}\frac{\partial p}{\partial X_j} \tag{68}$$

This is just a row of the inverse Jacobian! Like what is seen in equation 24

$$\sum_{j=1}^{3} \frac{\partial X_j}{\partial x_i}\frac{\partial p}{\partial X_j} = \sum_{j=1}^{3} \frac{1}{\det(\hat{\mathbf{J}})}\frac{\partial \det(\hat{\mathbf{J}})}{\partial(\frac{\partial x_i}{\partial X_j})}\frac{\partial p}{\partial X_j} \tag{69}$$

Then we can concise notation introducing a the matrix $\frac{\partial \det(\hat{\mathbf{J}})}{\partial(\frac{\partial x_i}{\partial X_j})} = \hat{\boldsymbol{\mathcal{J}}}_{i,j}$ Giving:

$$= \sum_{j=1}^{3} \frac{1}{\det(\hat{\mathbf{J}})}\hat{\boldsymbol{\mathcal{J}}}_{i,j}\frac{\partial p}{\partial X_j} \tag{70}$$

Combining this with the pressure results in the follow equation of motion as:

$$\rho_0 \frac{\partial^2 x_i(\mathbf{X}, t)}{\partial t^2} = -\sum_{j=1}^{3} \hat{\boldsymbol{\mathcal{J}}}_{i,j}\frac{\partial p}{\partial X_j} \tag{71}$$

The question now is, can we arrive at the same conclusion by minimizing the action of our system? The action can't be written exactly the same as the previous section, now we must also integrate over some domain R_0 containing our fluid element. Then similarly to previous variation restrictions, we have to assume the variation is 0 at R_0. One of our terms is obvious, it is simply the kinetic energy defined exactly as seen with particles.

$$\frac{\rho_0}{2}\sum_{i=1}^{3}\left(\frac{\partial x_i(\mathbf{X}, t)}{\partial t}\right)^2 \tag{72}$$

Whereas the potential energy is a bit more difficult to write. In the absence of an external field, the only potential energy a fluid would have is the internal energy U. To discuss the variation in U we can use the thermodynamic identity. If we assume that the entropy of a given element is constant with respect to the material co-ordinates we get the change in internal energy per unit mass as:

$$dU = TdS - pd\left(\frac{1}{\rho}\right) \tag{73}$$

If we assume the fluid acts like an ideal fluid then we know:

$$T = \frac{p}{R\rho} \tag{74}$$

$$U = \frac{1}{\gamma - 1}\frac{p}{\rho} \tag{75}$$

Where $\gamma = \frac{C_p}{C_v}$. If we then assume the gas is mono atomic then we know that $C_p = \frac{5}{2}R, C_v = \frac{3}{2}R$. This would give the familiar form:

$$U = \frac{3}{2}\frac{p}{\rho} = \frac{3}{2}RT \tag{76}$$

The reason we use the ratio of specific heats is that it allows for an extension to non-mono atomic fluids. Then we can write the pressure as:

$$p = \kappa\rho^{\gamma}e^{(\gamma-1)S/R} \tag{77}$$

These are derivable from the Sakur-Tetrode equation. This gives the action as:

$$\int_{t_1}^{t_2}\int_{R_0}\left(\frac{\rho_0}{2}\sum_{i=1}^{3}\left(\frac{\partial x_i(\mathbf{X},t)}{\partial t}\right)^2 - \rho_0 U\right)d\mathbf{X}dt \tag{78}$$

We can now take a variational derivative with respect to an infinitesimal change $\delta\mathbf{x}$ of the path $\mathbf{x}(\mathbf{X},t)$ of the fluid element $\mathbf{X}$. Such that at the boundary of R_0 the variation is zero. Adding the small variation in path $\epsilon\boldsymbol{\eta}(\mathbf{X},t)$ we get:

$$\int_{t_1}^{t_2}\int_{R_0}\left(\frac{\rho_0}{2}\sum_{i=1}^{3}\left(\frac{\partial(x_i(\mathbf{X},t)+\epsilon\eta_i(\mathbf{X},t))}{\partial t}\right)^2 - \rho_0 U(\mathbf{x}(\mathbf{X},t)+\epsilon\boldsymbol{\eta}(\mathbf{X},t))\right)d\mathbf{X}dt \tag{79}$$

We can simplify this slightly:

$$\int_{t_1}^{t_2}\int_{R_0}\left(\frac{\rho_0}{2}\sum_{i=1}^{3}\left(\frac{\partial x_i(\mathbf{X},t)}{\partial t}+\epsilon\frac{\partial\eta_i(\mathbf{X},t)}{\partial t}\right)^2 - \rho_0 U(\mathbf{x}(\mathbf{X},t)+\epsilon\boldsymbol{\eta}(\mathbf{X},t))\right)d\mathbf{X}dt \tag{80}$$

Expanding the square:

$$\int_{t_1}^{t_2}\int_{R_0}\left(\sum_{i=1}^{3}\left[\frac{\rho_0}{2}\left(\frac{\partial x_i(\mathbf{X},t)}{\partial t}\right)^2 + \rho_0\epsilon\frac{\partial\eta_i(\mathbf{X},t)}{\partial t}\frac{\partial x_i(\mathbf{X},t)}{\partial t}\right]\right.$$
$$\left. -\rho U(\mathbf{x}(\mathbf{X},t)+\epsilon\boldsymbol{\eta}(\mathbf{X},t)) + \mathcal{O}(\epsilon^2)\right)d\mathbf{X}dt \tag{81}$$

Then we can take the difference between this and the unvaried version gives:

$$\int_{t_1}^{t_2}\int_{R_0}\left(\sum_{i=1}^{3}\left[\rho_0\epsilon\frac{\partial\eta_i(\mathbf{X},t)}{\partial t}\frac{\partial x_i(\mathbf{X},t)}{\partial t}\right]-\rho_0\Big(U(\mathbf{x}(\mathbf{X},t)+\epsilon\boldsymbol{\eta}(\mathbf{X},t))-U(\mathbf{x}(\mathbf{X},t))\Big)+\mathcal{O}(\epsilon^2)\right)d\mathbf{X}dt \tag{82}$$

We can do the same integration by parts trick we performed earlier. Remembering that the variation is 0 at the boundary. Then rewriting the product of the sum of η_i and x_i as a dot product we get:

$$\int_{t_1}^{t_2}\int_{R_0}\left(-\rho_0\epsilon\boldsymbol{\eta}(\mathbf{X},t)\cdot\frac{\partial^2\mathbf{x}(\mathbf{X},t)}{\partial t^2}-\rho_0\Big(U(\mathbf{x}(\mathbf{X},t)+\epsilon\boldsymbol{\eta}(\mathbf{X},t))-U(\mathbf{x}(\mathbf{X},t))\Big)+\mathcal{O}(\epsilon^2)\right)d\mathbf{X}dt \tag{83}$$

We can factor out $\epsilon\boldsymbol{\eta}$ and note that $\frac{1}{\boldsymbol{\eta}}$ is a vector with each term of eta inverted.

$$\int_{t_1}^{t_2}\int_{R_0}\epsilon\boldsymbol{\eta}(\mathbf{X},t)\cdot\left(-\rho_0\frac{\partial^2\mathbf{x}(\mathbf{X},t)}{\partial t^2}-\frac{\rho_0\Big(U(\mathbf{x}(\mathbf{X},t)+\epsilon\boldsymbol{\eta}(\mathbf{X},t))-U(\mathbf{x}(\mathbf{X},t))\Big)}{\epsilon\boldsymbol{\eta}(\mathbf{X},t)}+\mathcal{O}(\epsilon)\right)d\mathbf{X}dt \tag{84}$$

Multiplying $1/\epsilon$ and then taking the limit as epsilon goes to zero will simplify the internal energy portion. We are taking a spatial derivative, i.e. how the internal energy change as we move around in space. Giving:

$$\int_{t_1}^{t_2}\int_{R_0}\boldsymbol{\eta}(\mathbf{X},t)\cdot\left(-\rho_0\frac{\partial^2\mathbf{x}(\mathbf{X},t)}{\partial t^2}-\rho_0\nabla U\right)d\mathbf{X}dt \tag{85}$$

This can now allow us to identify the variational derivative as:

$$-\rho_0\frac{\partial^2\mathbf{x}(\mathbf{X},t)}{\partial t^2}-\rho_0\nabla U=0 \tag{86}$$

We can simplify ∇U as follows:

$$\nabla U\sim\frac{\partial U(\rho(\mathbf{X},t))}{\partial x_i}=\frac{\partial U}{\partial\rho}\frac{\partial\rho}{\partial x_i} \tag{87}$$

Utilizing our thermodynamic equation of state we can simplify the internal energy derivative.

$$\frac{\partial U}{\partial\rho}=\frac{1}{\gamma-1}\left(\frac{\frac{\partial p}{\partial\rho}}{\rho}-\frac{p}{\rho^2}\right)=\frac{p}{\rho^2} \tag{88}$$

The derivative of the density with respect to x_i can be found by writing the density in terms of the Jacobian.

$$\frac{\partial}{\partial x_i}\rho(\mathbf{X}) = \sum_{j=1}^{3} \frac{\partial \rho(\mathbf{X})}{\partial X_j}\frac{\partial X_j}{\partial x_i} = \sum_{j=1}^{3} \frac{\partial \rho}{\partial p}\frac{\partial p}{\partial X_j}\frac{\partial X_j}{\partial x_i} \tag{89}$$

We then need to find the derivative of the density with respect to the pressure. Remembering that the pressure is defined: $p = \kappa\rho^{\gamma}e^{(\gamma-1)S/R}$. Hence,

$$\rho = \left(\frac{p}{\kappa e^{(\gamma-1)S/R}}\right)^{1/\gamma} \tag{90}$$

$$\implies \frac{\partial \rho}{\partial p} = 1/\gamma\left(\frac{p}{\kappa e^{(\gamma-1)S/R}}\right)^{1/\gamma-1}\left(\frac{1}{\kappa e^{(\gamma-1)S/R}}\right) \tag{91}$$

We can put a copy of $\rho^{\gamma}/\rho^{\gamma}$ to simplify.

$$1/\gamma\left(\frac{p}{\kappa e^{(\gamma-1)S/R}}\right)^{1/\gamma-1}\left(\frac{\rho^{\gamma}}{\rho^{\gamma}\kappa e^{(\gamma-1)S/R}}\right) \tag{92}$$

$$= \frac{(\rho^{\gamma})^{1/\gamma-1}}{\gamma}\left(\frac{\rho^{\gamma}}{p}\right) = \frac{\rho}{\gamma p} \tag{93}$$

Combining the last few steps gives:

$$\frac{\partial U}{\partial \rho}\frac{\partial \rho}{\partial x_i} = \frac{p}{\rho^2}\sum_{j=1}^{3}\frac{\rho}{\gamma p}\frac{\partial p}{\partial X_j}\frac{\partial X_j}{\partial x_i} = \frac{1}{\gamma\rho}\sum_{j=1}^{3}\frac{\partial p}{\partial X_j}\frac{\partial X_j}{\partial x_i} \tag{94}$$

The summation term we have here is the same as the one we had earlier. This implies:

$$-\rho_0\frac{\partial^2 \mathbf{x}(\mathbf{X},t)}{\partial t^2} - \rho_0\frac{1}{\gamma\rho\det(\hat{\mathbf{J}})}\sum_{j=1}^{3}\hat{\boldsymbol{\mathcal{J}}}_{i,j}\frac{\partial p}{\partial X_j} = 0 \tag{95}$$

But the Jacobian times the density is just the original density. Finally giving us:[9]

$$\rho_0\frac{\partial^2 x_i(\mathbf{X},t)}{\partial t^2} = -\frac{1}{\gamma}\sum_{j=1}^{3}\hat{\boldsymbol{\mathcal{J}}}_{i,j}\frac{\partial p}{\partial X_j} \tag{96}$$

[9]Where I have opted to split the vector equation up into each dimension individually.

3.2 Eulerian Form

The equations of motion for a spacial Eulerian description are given by:

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{\rho}\frac{\partial p}{\partial x_i} \tag{97}$$

We also have the conservation of mass and entropy in this scheme.

$$\frac{\partial \rho}{\partial t} + \sum_{j=1}^{3} \frac{\partial \rho u_j}{\partial x_j} = 0 \tag{98}$$

$$\frac{\partial S}{\partial t} + \sum_{j=1}^{3} \frac{\partial S u_j}{\partial x_j} = 0 \tag{99}$$

We can then introduce a separate relation for the pressure:

$$p = p(\rho, S) \tag{100}$$

With the set of equations established, we can write down the action.

$$\int\int_R \left(\frac{1}{2}\rho \mathbf{u}^2 - \rho E(\rho, S)\right) d\mathbf{r}dt \tag{101}$$

Adding a small variation in the velocity field $\epsilon\boldsymbol{\beta}$ and then subtracting the original functional gives::

$$\int\int_R \left(\rho\epsilon \mathbf{u}\cdot\boldsymbol{\beta} + \mathcal{O}(\epsilon^2)\right) d\mathbf{r}dt \tag{102}$$

This allows us to identify the functional derivative and set it to zero. Giving:

$$\rho\mathbf{u} = \mathbf{0} \tag{103}$$

Which just says that the velocity is zero. Clearly this formulation is incorrect, we only obtain the trivial solution. In order to resolve this we must consider the entropy and mass conservation equations. Notably if both of these are conserved we can also see that $S\rho$ should also be conserved. This gives the new action as:

$$\int\int_R \left(\frac{1}{2}\rho \mathbf{u}^2 - \rho E(\rho, S)\right) + \phi\left(\frac{\partial \rho}{\partial t} + \sum_{j=1}^{3} \frac{\partial \rho u_j}{\partial x_j}\right) + \zeta\left(\frac{\partial S\rho}{\partial t} + \sum_{j=1}^{3} \frac{\partial S\rho u_j}{\partial x_j}\right) d\mathbf{r}dt \tag{104}$$

Varying this with respect to the velocity will give:

$$\int \int_R \rho\epsilon \mathbf{u} \cdot \boldsymbol{\beta} - \zeta\epsilon \sum_{j=1}^{3} \frac{\partial S\rho\beta_j}{\partial x_j} - \phi\epsilon \sum_{j=1}^{3} \frac{\partial \rho\beta_j}{\partial x_j} + \mathcal{O}(\epsilon^2) d\mathbf{r} d \tag{105}$$

Performing the product rule results in:

$$\int \int_R \rho\epsilon \mathbf{u} \cdot \boldsymbol{\beta} - \zeta\epsilon \sum_{j=1}^{3} \left(\beta_j \frac{\partial S\rho}{\partial x_j} + S\rho \frac{\partial \beta_j}{\partial x_j} \right) - \phi\epsilon \sum_{j=1}^{3} \left(\beta_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial \beta_j}{\partial x_j} \right) + \mathcal{O}(\epsilon^2) d\mathbf{r} dt \tag{106}$$

$$= \int \int_R \epsilon \boldsymbol{\beta} \cdot \left[\rho \mathbf{u} - \zeta \frac{\partial S\rho}{\partial \mathbf{x}} - \phi \frac{\partial \rho}{\partial \mathbf{x}} \right] - \epsilon \sum_{j=1}^{3} \left[\zeta S\rho \frac{\partial \beta_j}{\partial x_j} + \phi\rho \frac{\partial \beta_j}{\partial x_j} \right] + \mathcal{O}(\epsilon^2) d\mathbf{r} dt \tag{107}$$

We can now abuse the integration by parts trick yet again.

$$\int \int_R \epsilon \boldsymbol{\beta} \cdot \left[\rho \mathbf{u} - \zeta \frac{\partial S\rho}{\partial \mathbf{x}} - \phi \frac{\partial \rho}{\partial \mathbf{x}} + \frac{\partial \zeta S\rho}{\partial \mathbf{x}} + \frac{\partial \phi\rho}{\partial \mathbf{x}} \right] + \mathcal{O}(\epsilon^2) d\mathbf{r} dt \tag{108}$$

Expanding into the chain rule and then simplifying gives the following:

$$\int \int_R \epsilon \boldsymbol{\beta} \cdot \left[\rho \mathbf{u} + S\rho \frac{\partial \zeta}{\partial \mathbf{x}} + \rho \frac{\partial \phi}{\partial \mathbf{x}} \right] + \mathcal{O}(\epsilon^2) d\mathbf{r} dt \tag{109}$$

This allows us to identify the variational derivative as:

$$\rho \mathbf{u} + S\rho \frac{\partial \zeta}{\partial \mathbf{x}} + \rho \frac{\partial \phi}{\partial \mathbf{x}} = 0 \tag{110}$$

$$\implies \mathbf{u} = S \frac{\partial \zeta}{\partial \mathbf{x}} + \frac{\partial \phi}{\partial \mathbf{x}} \tag{111}$$

Suppose we restrict ourselves to systems with constant entropy throughout the system (i.e. homentropic). Then we can say that the velocity can be written in terms of some potential defined in terms of our Lagrange multipliers.

$$S \neq f(\mathbf{x}) \implies \mathbf{u} = \frac{\partial}{\partial \mathbf{x}} (S\zeta + \phi) \tag{112}$$

This is the exact assumption for a potential flow. Similarly, this implies that if we have a homentropic system that the flow must be irrotational. This indicates that the vorticity and entropy are connected in some manner. This can also point to the fact that highly turbulent flows, hence those with highly local

vorticity, will produce entropy gradients in the system. However, while potential flow can be useful, it is not what we are looking for. Generally potential flow is not applicable to real world fluids. For example, d'Alembert's paradox shows that there is a lack of drag and lift forces in potential flow.

Similarly, this also implies that all vorticity is due to some type of entropy gradient. This should not always be the case. Considering this error, we can include another informed constraint. A key difference between the original principle of least action and the current development is the idea of indistinguishably. When you use classical mechanics, it is as if each particle is painted a specific unique color. This means if I were to mix them up, you would be able to tell the difference. However, when you are dealing with Eulerian fluid phenomena you don't have such a luxury. You opt to not care about the exact particle at a given point in space, rather you care about the observable quantities there. This is the key cause of difficulty. We can create a new quantity denoted $\boldsymbol{\alpha}$. This quantity is the particle labels. You could generate this by considering a fluid at a given time $t = 0$, and then marking the vector position of every fluid element. In a sense then alpha is just the material co-ordinates $\boldsymbol{\alpha} = \mathbf{X}$ we dealt with in the Lagrangian formulation of the action. Supposing we could identify the exact particle at a given point in space would essentially be saying that $\boldsymbol{\alpha}$ is conserved. This is the Lin constraint, and we denote it's conservation as follows:

$$\frac{\partial \rho \boldsymbol{\alpha}}{\partial t} + \sum_{j=1}^{3} \frac{\partial \rho u_j \boldsymbol{\alpha}}{\partial x_j} = 0 \tag{113}$$

Including this constraint with the Lagrange multiplier λ will result in the following after varying the velocity:

$$\mathbf{u} = S\frac{\partial \zeta}{\partial \mathbf{x}} + \boldsymbol{\alpha}\frac{\partial \lambda}{\partial \mathbf{x}} + \frac{\partial \phi}{\partial \mathbf{x}} \tag{114}$$

This setup can be made even more interesting if we vary with respect to other functions, such as the density, entropy, and material co-ordinate. We start again with the action functional with the Lagrange multipliers to enforce conservation. We wish to now vary the action w.r.t. to the density. Ideally this should result in appropriate law for the how the density should act. The variation is going to be denoted as $\epsilon\tau$, where epsilon will be taken to zero eventually and tau is a function of space. Rather than walk through the steps directly, I will explain the key steps. First, apply the variation $\epsilon\tau$, then subtract the unvaried quantity. Then apply the integration by parts trick and reduce the expression. It should be fairly easy to pick out the variation times something in the integrand. This should result in the following property:

$$\delta\rho \implies \frac{1}{2}\rho\mathbf{u}^2 - H = \frac{D\phi}{Dt} + \frac{D\zeta}{Dt} + \frac{D\lambda}{Dt} \tag{115}$$

Where H is the enthalpy is defined in terms of the pressure:

$$H = E + p/\rho \tag{116}$$

Listing off a two other results from taking a variation we can see:

$$\delta S \implies T = -\frac{D\zeta}{Dt} \tag{117}$$

$$\delta \boldsymbol{\alpha} \implies 0 = \frac{D\lambda}{Dt} \tag{118}$$

What do these conditions tell us about a fluid where the material co-ordinate is conserved? Consider the vorticity.

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \nabla S \times \nabla \zeta + \nabla \boldsymbol{\alpha} \times \nabla \lambda \tag{119}$$

This allows for the source of rotation to be due to quantities other than the entropy. As Seliger points out, this could be due to viscous forces.

We can simplify this principle significantly by integrating the Lagrangian with constraints by parts. Due to the surface term being zero it clear that:

$$\int \int_R \left(\frac{1}{2}\rho \mathbf{u}^2 - \rho E(\rho, S) \right) + \phi \left(\frac{\partial \rho}{\partial t} + \sum_{j=1}^{3} \frac{\partial \rho u_j}{\partial x_j} \right) \tag{120}$$

$$+ \zeta \left(\frac{\partial S\rho}{\partial t} + \sum_{j=1}^{3} \frac{\partial S\rho u_j}{\partial x_j} \right) + \lambda \left(\frac{\partial \boldsymbol{\alpha}\rho}{\partial t} + \sum_{j=1}^{3} \frac{\partial \boldsymbol{\alpha}\rho u_j}{\partial x_j} \right) d\mathbf{r}dt \tag{121}$$

$$= \int \int_R \frac{1}{2}\rho \mathbf{u}^2 - \rho E(\rho, S) - \rho \frac{D\phi}{Dt} - \rho S \frac{D\zeta}{Dt} - \rho \boldsymbol{\alpha} \frac{D\lambda}{Dt} d\mathbf{r}dt \tag{122}$$

Using equation 115 we can remove the need for the material derivatives and write:

$$= \int \int_R \frac{1}{2}\rho \mathbf{u}^2 - \rho E(\rho, S) - \rho \frac{D\phi}{Dt} - \rho S \frac{D\zeta}{Dt} - \rho \boldsymbol{\alpha} \frac{D\lambda}{Dt} d\mathbf{r}dt \tag{123}$$

$$= \int \int_R \rho H - \rho E(\rho, S) d\mathbf{r}dt = \int \int_R p(H, S) d\mathbf{r}dt \tag{124}$$

Showing that the Lagrangian density is just the pressure! Implying we just need to variationally minimize the pressure over our domain. While these results are just reformulations of the action in different scenarios, it would be interesting to apply this technique to derive velocity fields of our fluid.

4 Examples of Variational Calculus for Fluids

4.1 Shear Flow

In two dimensions a shear flow can be parameterized by:

$$u_1 = u_1(y), u_2 = 0 \tag{125}$$
$$\rho = \rho(y), S = S(y) \tag{126}$$

This is because the velocity field should only be a function of the vertical coordinate y. Similarly, this would imply that the enthalpy and the temperature should be functions of y alone. Intuitively this happens because we have symmetry as you translate in the x-direction. Mathematically we can see this due to the relations:

$$H = E + \frac{p}{\rho} \tag{127}$$

4.2 Maxwell's Equations

Electricity and magnetism has many parallels between it and fluid mechanics. Rather than directly using the results from the previous sections we will be taking some of the ideas of variational principles on continuums and applying them on $\mathbf{B}$ and $\mathbf{E}$. In the past most solutions for electromagnetic waves are described by Maxwell's equations where the charge density is zero. These are given by:

$$\nabla \cdot \mathbf{E} = 0 \tag{128}$$
$$\frac{\partial \mathbf{E}}{\partial t} - c^2 \nabla \times \mathbf{B} = 0 \tag{129}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{130}$$
$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{131}$$

If we think of the electromagnetic field as a harmonic oscillator we can designate the magnetic field as a sort of "kinetic" energy and the electric field as a "potential" energy.

$$L = \frac{\epsilon_0}{2}(c^2 \mathbf{B}^2 - \mathbf{E}^2) \tag{132}$$

As a sanity check we can see that the units of ϵ_0 are $\left[\frac{\text{Coulomb}^2}{\text{Newton}\cdot\text{Meter}^2}\right]$ and the units of $\mathbf{E}$ are $\left[\frac{\text{Newton}}{\text{Coulomb}}\right]$. This gives the units of the term involving the electric field as $\left[\frac{\text{Newton}}{\text{Meter}^2}\right]$. Similarly, because $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ we get $\mathbf{B}^2/\mu_0$ which would

have units of $\left[\frac{\text{Newton}}{\text{Meter}^2}\right]$ as well. When this is integrated over space we get units corresponding to energy, as expected for a Lagrangian.

Taking some ideas from the previous section, if we want to variationally minimize this without finding trivial solutions we need some side conditions. We can show the need for this by finding the variation on the Lagrangian alone. The action will be:

$$S = \int \int_R \left(\frac{\epsilon_0}{2} (c^2 \mathbf{B}^2 - \mathbf{E}^2) \right) d\mathbf{r} dt \tag{133}$$

Adding a small variation $\epsilon \boldsymbol{\beta}$ to $\mathbf{B}$ gives:

$$\int \int_R \left(\frac{\epsilon_0}{2} (c^2 (\mathbf{B} + \epsilon \boldsymbol{\beta})^2 - \mathbf{E}^2) \right) d\mathbf{r} dt \tag{134}$$

Which reduces to this when you subtract the original functional:

$$\int \int_R \left(\epsilon_0 c^2 \epsilon \mathbf{B} \cdot \boldsymbol{\beta} + \mathcal{O}(\epsilon^2) \right) d\mathbf{r} dt \tag{135}$$

By then dividing by ϵ the functional derivative is easily identified as:

$$\frac{\delta S}{\delta \mathbf{B}} = \epsilon_0 c^2 \mathbf{B} \tag{136}$$

Which results in the trivial case of $\mathbf{B} = 0$ at the critical point. The electric field is shown to be zero similarly. In order to fix this we need constraints on our Lagrangian. We can choose two of Maxwell's equations and conserve them using Lagrange multipliers. Our choice of equations will affect our answer for each field.

Choosing 128 and 130 to be conserved will result in both fields being described by a scalar potential. However, this would simply make the situation become static. We can consider the Poynting vector $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$. This represents the energy flux density of the EM field. If both fields are scalar potentials then that implies $\nabla \cdot \mathbf{S} = 0$, meaning that there will be no time derivative of the energy density[10] in the region R.

Suppose we choose equations 129 and 130.

4.3 Plasma Waves

5 Bibliography

Please check out the paper by Seliger and Whitham. It was instrumental in my understanding of using variational calculus as well as how to apply it in

[10] A stack exchange showing the mathematical reason for this can be found here. Here is another stack exchange that discusses when the Poynting vector is zero. In general this point of choosing which of Maxwell's to conserve is not fully understood in my mind. This certainly deserves some more exploration and discussion. Please let me know if this interests you as well!

continuum mechanics. This documents last two sections are just a reproduction of their results with some extra steps filled out for my own understanding. Feel free to email me if you have questions, find a typo in my work, or you want to roast my notation. You can find my contact information on my website `https://winstonsullivan.netlify.app/`

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