Quantum Mechanics

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1 The Axioms of Quantum

1.1 Axiom 1:

> Behind every quantum system there is an associated separable complex Hilbert space $\mathcal{H}$. The states of the system are all given by: <u>positive</u> <u>trace class</u> <u>linear</u> maps $\rho : \mathcal{H} \to \mathcal{H}$. Where the $\mathrm{Tr}\rho = 1$.

A system described by a complex Hilbert space can be thought of as tuple of 4 items $\Big(\mathcal{H}, +, \cdot, \langle\cdot|\cdot\rangle\Big)$

- $\mathcal{H}$ is a set
- $+$ is a map $+ : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$
- $\cdot$ is a map $\cdot : \mathbb{C} \times \mathcal{H} \to \mathcal{H}$
- $\langle\cdot|\cdot\rangle$ is the sesqui-linear product, or map $\langle\cdot|\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$.

With the basics of these maps outlined, the Hilbert space has three main qualities. (1) If we look at just the first three elements of the tuple that defines our Hilbert space $\big(\underline{\mathcal{H}, +, \cdot}, \langle\cdot|\cdot\rangle\big)$ they must satisfy the axioms of a vector space CANI ADDU. Namely, Take three elements ψ, φ, and α all of which are $\in \mathcal{H}$ and $z, x \in \mathbb{C}$. CANI and ADDU are:

For addition

- Commutative: $\varphi + \psi = \psi + \varphi$
- Associative: $(\varphi + \psi) + \alpha = \psi + (\varphi + \alpha)$
- Neutral: $\psi + \mathbf{0} = \psi$
- Inverse: $\psi + (-\psi) = 0$

For scalar multiplication

- Associative: $z(y\psi) = y(z\psi)$

- Distributive 1: $(x+z)\psi = x\psi + z\psi)$
- Distributive 2: $x(\psi+\varphi) = x\psi + x\varphi$
- Unit Element: $1\psi = \psi$

(2) The sesqui-linear product must have the following properties:

(i) $\langle\psi|\varphi\rangle = \overline{\langle\varphi|\psi\rangle}$. By exchanging the ordering it gives us the complex conjugate of the original ordering.

(ii) $\langle\psi|z\varphi_1 + \varphi_2\rangle = z\langle\psi|\varphi_1\rangle + \langle\psi|\varphi_2\rangle$. Linearity in the second argument.

(iii) $\langle\psi|\psi\rangle \geq 0$. Where the equality only holds when $\psi = 0_{\mathcal{H}}$

for any $\psi, \varphi_1, \varphi_2 \in \mathcal{H}$ and $z \in \mathbb{C}$ (Note this is also called the inner product which I will be using despite it not being as precise of a name).

Lastly, (3) The Hilbert space must be complete. This aspect is a bit more complex. If we take a sequence in $\mathcal{H}, \phi : \mathbb{N} \to \mathcal{H}$, it must satisfy the Cauchy property: For all epsilon greater than zero, there exists an N in the positive integers such that for all n and m greater than N it is true that the norm of the nth member minus the mth member is less than epsilon. (Note that the norm is the square root of the inner product. We will go into some more details on this soon)

$$\forall\epsilon > 0\ \exists N \in \mathbb{N}\ \forall n, m > N : ||\phi_n - \phi_m|| < \epsilon \tag{1}$$

This is basically a weak convergence statement. So, if you have a sequence in $\mathcal{H}$ you can already conclude that it converges to an element of $\mathcal{H}$.

It is time for a brief rant on states that is partly Schuller and partly from a conversation between me, Brenda (as usual) and Mike Swag. Often people label the element ψ like this $|\psi\rangle$. This is called a "ket" and the dual element $\langle\psi|$ is called a "bra". This naming convention comes from Dirac and it directly follows from the Riesz lemma. It is often stated that the state of a quantum mechanical system can be denoted by a ket $\psi \in \mathcal{H}$. This is false. We can only represent a state with a ket if it is a pure state (if the quantum state of a system is not given by a pure state it is called a mixed state). The pure states are good enough when discussing systems where perfect information about the state is known. For systems where pure states are sufficient, the set of pure states can be thought of as a unit sphere around the origin. This contrasts the case of a state in classical mechanics. Rather than the state being characterized by its canonical momentum and position it is an operator on an element in $\mathcal{H}$. Mathematically pure states satisfy the following: There exists a psi in the Hilbert space such that for all phi in the Hilbert space, the pure state rho acts on phi by taking the inner product between psi and phi. It is then normalized by the inner product between psi and itself which is all then multiplied by psi.

$$\exists\psi \in \mathcal{H} : \forall\varphi \in \mathcal{H} : \rho(\varphi) = \frac{\langle\psi,\varphi\rangle}{\langle\psi,\psi\rangle}\psi \tag{2}$$

It is important to note at this point, for most QM systems $\mathcal{H}$ is infinite dimensional. However there are certain systems where a finite dimensional Hilbert space is fine. In these cases we can represent a ket by a vector with indices like [a,b,c] la la la. It is very important to note that a ket can be represented by a vector IF AND ONLY IF the state is finite dimensional. With this in mind, to shed some light on what a pure state really is consider a finite dimensional Hilbert space $\mathcal{H}$ where φ and ψ are elements of $\mathcal{H}$. Now consider the pure state ρ defined by the equation above. Recall the vector projection operation from multi-variable calculus:

$$\begin{aligned}
\text{Proj}_{\psi}\alpha &= |\vec{\varphi}|\cos(\theta)\hat{\psi} &(3)\\
&= \frac{|\vec{\varphi}|\cos(\theta)}{|\vec{\psi}|}\vec{\psi} &(4)\\
&= \frac{|\vec{\varphi}||\vec{\psi}|\cos(\theta)}{|\vec{\psi}|^2}\vec{\psi} &(5)\\
&= \frac{\vec{\varphi}\cdot\vec{\psi}}{\vec{\psi}\cdot\vec{\psi}}\vec{\psi} &(6)
\end{aligned}$$

By reorganizing the form it beckons the interpretation that pure states amount to a projection of α into the direction of ψ (dot product in regular calculus is its version the inner product). We could represent a projection by just a vector in the direction of the ψ, but this does not account for the variable length of the outputted vector. Therefore it is incorrect to say a pure state is simply a vector in $\mathcal{H}$, due to the lack of consideration for every vector that points in the same direction. A ray is a more accurate representation of a pure state. Back to the axioms :).

We have dissected what a Hilbert space but we have yet to touch on the particular aspects of a state. Firstly lets look at the linear portion. What is a linear map? A linear map $\mathbf{A}$ is defined as follows:

$$\begin{aligned}
&\mathbf{A} : \mathcal{D}_A \to \mathcal{H} &(7)\\
&\text{Such that: } \mathbf{A}(\psi + \alpha\varphi) = \mathbf{A}(\psi) + \alpha\mathbf{A}(\varphi) &(8)\\
&\forall\psi, \varphi \in \mathcal{H} \text{ and } \alpha \in \mathbb{C} &(9)
\end{aligned}$$

Now we can abuse notation a bit and simply use $\mathbf{A}\psi$ rather than $\mathbf{A}(\psi)$, but it is recognized as A acts on psi because A is a map. Also note, when I use $\mathcal{D}_A$ I am saying that it is densely defined W.R.T the Hilbert space. Essentially taking the Cauchy property from before but saying that for all epsilon ¿ 0 and for all elements of the Hilbert space there exists an element in the domain of A such that the normed difference of each element is less than epsilon. Now, what is a positive map? Using A from above, It's defined as follows:

$$\forall \psi \in \mathcal{D}_A : \langle \psi, \mathbf{A}\psi \rangle \geq 0 \tag{10}$$

And lastly the Trace Class aspect. For $\mathbf{A} : \mathcal{H} \to \mathcal{H}$ to be of Trace Class it must satisfy:

$$\text{for any orthonormal basis } \{e_n\} \text{ of } \mathcal{H} \tag{11}$$

$$\sum_{e_n} \langle e_n, \mathbf{A}e_n \rangle < \infty \tag{12}$$

Noting that this map must have the entire Hilbert space as the domain in order to evaluate the summation/series. ($\mathrm{Tr}\mathbf{A} := \sum_{e_n} \langle e_n, \mathbf{A}e_n \rangle$)

1.2 Axiom 2

> The observables of a quantum system are the self-adjoint linear maps $\mathbf{A} : \mathcal{D}_A \to \mathcal{H}$

That's it! But, what is self adjointness (It sure as shit isn't Hermicity and isn't symmetric)? It is simple for finite dimensional $\mathcal{H}$ but not so for infinite ones. A map $\mathbf{A}$ is self-adjoint if:

$$\mathbf{A} : \mathcal{D}_A \to \mathcal{H} \tag{13}$$

$$\text{it coincides with its adjoint map } \mathbf{A}^* \tag{14}$$

$$\mathbf{A}^* : \mathcal{D}_{A^*} \to \mathcal{H} \tag{15}$$

$$\text{where coincides means } \mathcal{D}_{A^*} = \mathcal{D}_A \text{ and } \mathbf{A}^*\psi = \mathbf{A}\psi \ \forall \psi \in \mathcal{D}_{A^*} \text{ or } \mathcal{D}_A \tag{16}$$

It is extra important to consider the domains. When you don't you can get wacky conflicting answers (*COUGH COUGH INFINITE SQUARE WELL COUGH COUGH*). Okay cool, but what is the adjoint??? A map $\mathbf{A}$s adjoint $\mathbf{A}^*$ is defined by its domain. Namely,

$$\mathbf{A} : \mathcal{D}_A \to \mathcal{H} \tag{17}$$

$$\mathbf{A}^* : \mathcal{D}_{A^*} \to \mathcal{H} \tag{18}$$

$$(i)\ \mathcal{D}_{A^*} := \{\psi \in \mathcal{H} \,\Big|\, \forall \alpha \in \mathcal{D}_A \exists \eta \in \mathcal{H} : \langle \psi, \mathbf{A}(\alpha) \rangle = \langle \eta, \alpha \rangle\} \tag{19}$$

$$(ii)\ \text{Where } \mathbf{A}^*(\psi) := \eta \tag{20}$$

If you are reading this and trying to learn, don't be worried if it all seems wack and not what Mr. Griffiths told you. It will all come eventually, just watch the quantum god Schuller do his work. He's got u.

1.3 Axiom 3

The probability that a measurement of an observable **A** on a system that is in the state ρ yields a result in the Borel set $E \subseteq \mathbb{R}$ is given by:

$$\mu_\rho^A(E) := \mathrm{Tr}(P_A(E) \circ \rho)$$
$$\text{Where } P_A : \mathrm{Borel}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$$

is the unique projection valued measure that is associated with a self-adjoint map **A** according to the spectral theorem:

$$A = \int_{-\infty}^{\infty} \lambda dP_A(\lambda)$$

Wow, that is a lot to unpack. First, mu is the measure associated with the state ρ and the observable **A** applied to the measurable set E. Where the $\circ$ refers the composition of the two linear maps. We will see that P_A is bounded and that the composition of a bounded map with one that is of trace class is again trace class. Terribly technical yet if you don't know you will start doing nonsense - Schulla. Now, $\mathcal{L}(\mathcal{H})$ is the Banach space of bounded linear maps on $\mathcal{H}$ and the Borel set will be given in more detail later, it relates to the sets on which you can meaningfully assign a volume (Operator = Linear Map). Projection valued measures will be touched on later and we will climax on the first half of this course with the spectral theorem. Generally this axiom is pretty non-understandable without some work. Time to grind so we can do step 1!!!! These first three axioms are the "kinematic" axioms. Telling us about the objects you use to talk about the systems. On to "dynamics", how a QM system develops in time.

1.4 Axiom 4

Unitary Dynamics: during time intervals (t_1, t_2) where no measurement occurs, the state $\rho(t_1)$@ time $= t_1$ is related to $\rho(t_2)$@ time $= t_2$ through the relaton:

$$\rho(t_2) = \mathbf{U}(t_2 - t_1)\rho(t_1)\mathbf{U}(t_2 - t_1)^{-1}$$
$$\text{Where } \mathbf{U}(t) := \exp(-i\mathbf{H}t/\hbar) \text{ such that } \mathbf{H} \iff \text{Energy}$$

In order to even talk about time evolution we need to discuss what a function of an infinite dimensional operator is. For scalars it makes sense, even for finite matrices it does (use the Taylor expansion), but what the hell does it mean for infinite dimensions. For that we again require the spectral theorem, hence referencing it twice inside the axioms alone. We need that spectral theorem baby!!!

1.5 Axiom 5

Projective Dynamics: When a measurement is made at a time t_m, then the state ρ_{after} immediately after the measurement of an observable $\mathbf{A}$ is

$$\rho_{\text{after}} := \frac{P_A(E)\rho_{\text{before}}P_A(E)}{\text{Tr}(P_A(E)\rho_{\text{before}}P_A(E))} \tag{21}$$

$$\tag{22}$$

Where E is the smallest Borel set where the actual outcome of the measurement happened to lie.

This is the wavefunction collapse. Projective dynamics are extra annoying because they depend on the actual measurement that has occurred and it requires us to know the result of the measurement. Implying that we can not predict the outcome of an experiment. Bringing us to the end of lecture 1. It is a lot to unpack, don't get too stressed. But also, as I have learned, dont get too confident. As Richard Feynman says, no one understands quantum mechanics. (But that isnt going to stop us from trying!!!! Physics is futile, get over it)

2 Banach Spaces

A Banach space is a generalization of a Hilbert space. They often appear in constructions that start with a Hilbert space.

A Banach space is a vector space with a norm, which is complete with respect to this norm. $(V, +, \cdot, ||\cdot||)$.Such that the norm satisfies the following: $\forall \lambda \in \mathbb{C}$ and $f \in V$

$$\text{(i) } ||\lambda f|| = |\lambda|\ ||f|| \tag{23}$$

$$\text{(ii) } ||f+g|| \leq ||f|| + ||g|| \tag{24}$$

$$\text{(iii) } ||f|| \geq 0, = 0 \text{ iff } f = 0_V \tag{25}$$

2.1 Maps or Operators

It is often beneficial to study maps between spaces. If we restrict ourselves to maps that conserve particular quantities than we can make claims about the spaces in which they are mapping. To start, consider the map $\mathbf{A}$

$$\mathbf{A} : V \to W \tag{26}$$

$$\text{Such that: V is a normed space } (V, +, \cdot, ||\cdot||_V) \tag{27}$$

$$\text{and W is a Banach space } (W, +, \cdot, ||\cdot||_W) \tag{28}$$

Considering the map above, a map $\mathbf{A}$ is said to be <u>bounded</u> if:

$$\sup_{f \in V} \frac{||\mathbf{A}f||_W}{||f||_V} < \infty \tag{29}$$

This can be rewritten as

$$\sup_{||f||_V = 1} ||\mathbf{A}f||_W < \infty \tag{30}$$

By considering only the ones that have norm 1, we can use the (i)th property of the norm to adjust the first definition into the second one. We can then use this boundedness condition to define the norm of an operator/map.

$$||\mathbf{A}|| := \sup_{f \in V} \frac{||\mathbf{A}f||_W}{||f||_V} \text{ if } \mathbf{A} \text{ is bounded.} \tag{31}$$

Bounded operators are important in quantum mechanics due to their ease of use. Often we will deal with unbounded ones but we end up reducing the problem into one that deals with a sequence of bounded operators. Unbounded operators don't have a norm like the one defined above. Interestingly enough, both the operators corresponding to momentum and position are unbounded. Unbounded operators can not be commuted, implying that the usual Heisenberg commutation relation can not be written.

2.2 Examples

As an example lets consider the identity map.

$$V = W = \text{ Banach} \tag{32}$$

$$\mathbf{I}_W : W \to W \tag{33}$$

$$f \mapsto f \tag{34}$$

We can then check to see if the identity operator is bounded, only then can we calculate its operator norm.

$$\sup_{||f||_W = 1} ||\mathbf{I}_W f||_W = \sup_{||f||_W = 1} ||f||_W = 1 < \infty \tag{35}$$

$$\implies ||\mathbf{I}_W|| = 1 \tag{36}$$

As a counterexample, we can consider a different Banach space and a different normed space. Consider the set of continuous functions that map an element on [0,1] to a number in the complex plane.

$$W = C^0([0,1]) = \{f : [0,1] \to \mathbb{C} \big| f \text{ is continuous}\} \tag{37}$$

To be specific, we can define a continuous map by considering an open sets. If the pre-image of any open set in the target space is open then the map is continuous. To make a Banach space we also require a norm. Define the norm as:

$$||f||_\infty := \sup_{x\in[0,1]} |f(x)| \tag{38}$$

Now what remains is to show that $(W, ||f||_\infty)$ makes a vector space that is complete. Only then will we make a Banach space. Consider two elements f and g along with the scalar λ. Addition in our space is defined as: $W \times W \mapsto W \quad (f +_W g) := f(x) +_{\mathbb{C}} g(x)$. Scalar multiplication is defined as: $W \times \mathbb{C} \mapsto W \quad \lambda \cdot_W f := \lambda \cdot_{\mathbb{C}} f(x)$. CANI and ADDU are shown as follows:

For addition

- Commutative: ✓

$$f +_W g = f(x) +_{\mathbb{C}} g(x) = g(x) +_{\mathbb{C}} f(x) = g +_W f \tag{39}$$

- Associative: ✓

$$(f +_w g) +_w h = (f(x) +_{\mathbb{C}} g(x)) +_{\mathbb{C}} h(x) \tag{40}$$
$$= f(x) +_{\mathbb{C}} (g(x) +_{\mathbb{C}} h(x)) = f +_W +(g_W + h_W) \tag{41}$$

- Neutral: ✓

$$\mathbf{0} : \forall x \in [0,1] : \mathbf{0}(x) \mapsto 0 \tag{42}$$
$$\implies f +_W \mathbf{0} = f(x) +_{\mathbb{C}} \mathbf{0}(x) = f(x) + 0 = f \tag{43}$$

- Inverse:

$$-f : \forall x \in [0,1] : -f(x) \mapsto 0 \tag{44}$$
$$\implies f +_W \mathbf{0} = f(x) +_{\mathbb{C}} \mathbf{0}(x) = f(x) + 0 = f \tag{45}$$

For scalar multiplication

- Associative: $z(y\psi) = y(z\psi)$
- Distributive 1: $(x + z)\psi = x\psi + z\psi)$
- Distributive 2: $x(\psi + \varphi) = x\psi + x\varphi$
- Unit Element: $1\psi = \psi$

Next we need to show that the norm satisfies the the three properties given above:
Lastly (the hardest one), we need to show completeness. We will do so in 4 steps.

1. Given a Cauchy sequence $\{f_n\}$ in W, show that a particular element $x_i \in [0,1]$ when mapped by $\{f_n\}$ forms a Cauchy sequence in $\mathbb{C}$.

   Since $\{f_n\}$ is Cauchy, we know the following:

   $$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall n, m > N : ||f_n - f_m||_\infty < \epsilon \tag{46}$$

   We can rewrite the infinity norm by its definition giving the following inequality:

   $$\sup_{x \in [0,1]} |f_n(x) - f_m(x)|_{\mathbb{C}} < \epsilon \tag{47}$$

   Since we are taking the supremum of elements x, we know that if it holds for the largest of the elements it holds for any arbitrary element x_i. This lets us put another inequality on the left of the supremum.

   $$|f_n(x_i) - f_m(x_i)|_{\mathbb{C}} < \sup_{x \in [0,1]} |f_n(x) - f_m(x)|_{\mathbb{C}} < \epsilon \tag{48}$$

   This implies

   $$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall n, m > N : |f_n(x_i) - f_m(x_i)|_{\mathbb{C}} < \epsilon \tag{49}$$

   Which is the definition of a Cauchy sequence in $\mathbb{C}$

2. Show that the resulting limit of the Cauchy sequence $\lim_{n\to\infty} f_n(x_i) = f(x_i)$ lies in $\mathbb{C}$.

   Because the complex numbers are complete we know that any Cauchy sequence will converge in $\mathbb{C}$. This allows us to define the limiting value.

   $$f(x_i) := \lim_{n\to\infty} f_n(x_i) \tag{50}$$

3. Show that the function $f(x)$ (where f is the limit from before) is continuous.

A map f is continuous if the pre-image of every open set in the target space is open. Mathematically this means

$$
\begin{aligned}
& f \text{ is cont. @ } f(y) \text{ iff.} && (51)\\
& \forall \epsilon > 0 : \exists \delta > 0 : \forall x : |x - y| < \delta && (52)\\
& \implies |f(x) - f(y)| < \epsilon && (53)
\end{aligned}
$$

If we consider a Cauchy sequence $\{f_n\}$ we know that $||f_n - f_m|| < \epsilon$ holds for all n and m greater than N. This implies we can just pick a particular m as long as its $> N$. Suppose $m = N + 69420$. This implies the following:

$$
\text{Given: } \forall \epsilon > 0 \; \exists N \in \mathbb{N} : \; \forall n, m > N : ||f_n - f_{N+69420}||_\infty < \epsilon \qquad (54)
$$

The infinity norm is certainly greater than the norm of f at a particular x_i. Allowing this sequence of inequalities.

$$
|f_n(x_i) - f_{N+69420}(x_i)|_{\mathbb{C}} < ||f_n - f_{N+69420}||_\infty < \epsilon \qquad (55)
$$

We can then take the limit as n goes to infinity.

$$
\lim_{n\to\infty} |f_n(x_i) - f_{N+69420}(x_i)|_{\mathbb{C}} = |f(x_i) - f_{N+69420}(x_i)|_{\mathbb{C}} < \epsilon \qquad (56)
$$

(Note that this limit only holds due to the continuity of the norm, it may not look like it but we have moved the limit inside the inner product) Due to the properties of the norm, we also know: $|f_{N+69420}(x_i) - f(x_i)|_{\mathbb{C}} < \epsilon$. We are going to use this fact later. Now let us remember that $f_{N+69420}$ is an element of W. This means it is continuous. All continuous functions satisfy the soft ball delta epsilon relation above. This means:

$$
\begin{aligned}
& \forall \epsilon > 0 : \exists \delta > 0 : \forall x : |x - y| < \delta && (57)\\
& \implies |f_{N+69420}(x) - f_{N+69420}(y)| < \epsilon && (58)
\end{aligned}
$$

With these three inequalities let us consider the difference between two inputs x and y of f.

$$
|f(x) - f(y)| = |f(x) - f_{N+69420}(x) + f_{N+69420}(x) - f_{N+69420}(y) + f_{N+69420}(y) - f(y)| \qquad (59)
$$

By the triangle inequality of the norm we know the following:

$$|f(x) - f_{N+69420}(x) + f_{N+69420}(x) - f_{N+69420}(y) + f_{N+69420}(y) - f(y)| \tag{60}$$

$$< |f(x) - f_{N+69420}(x)| + |f_{N+69420}(x) - f_{N+69420}(y)| + |f_{N+69420}(y) - f(y)| \tag{61}$$

As shown above, all three of these are $< \epsilon$ implying the following:

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x : |x - y| < \delta \tag{62}$$

$$\implies |f(x) - f(y)| < \epsilon \tag{63}$$

Which is the definition of continuity, meaning f is continuous.

4. Show how the sequence of $\{f_n\}$ converges to f

   As usual, we begin at the coochie sequence.

$$\text{Given: } \forall \epsilon > 0 \ \exists N \in \mathbb{N} : \ \forall n, m > N : ||f_n - f_m||_\infty < \epsilon \tag{64}$$

We can do the same trick and make a weaker statement by not using the supremum inside the infinity norm and then taking a limit of the one side.

$$|f_n(x) - f_m(x)| \leq ||f_n - f_m||_\infty < \epsilon \tag{65}$$

$$\implies |f(x) - f_m(x)| < \epsilon \tag{66}$$

$|f(x) - f_m(x)|$ is certainly less than or equal to the situation were we maximize this norm wrt to x.

$$|f(x) - f_m(x)| < \max_{x \in [0,1]} |f(x) - f_m(x)| < \epsilon \tag{67}$$

But that is just the supremum norm.

$$\implies \forall \epsilon > 0 \ \exists N \in \mathbb{N} : \ \forall m > N : ||f - f_m||_\infty < \epsilon \tag{68}$$

Which shows that the limit of the sequence indeed approaches f. QED.

With our Banach space $W = C^0([0,1])$ we can consider our normed space to be $V = C^1([0,1])$. A fact about V is that $C^1([0,1]) \subset C^0([0,1])$. It is the space of continuous functions such that their derivative is still continuous (The once continuously differentiable functions). It is important to show that it is a subvector space with the norm inherited from W. However we don't quite care if V is Banach. We can define an operator $\mathbf{P}$.

$$\mathbf{P} : V \to W \tag{69}$$

$$f \mapsto f' \tag{70}$$

This is the classic momentum operator with some scaling differences. Namely, there is no $-i\hbar$. It is clear that $\mathbf{P}$ is linear because it is the derivative. Finally we can get to the important part. Is $\mathbf{P}$ bounded? Therefore the claim is: $\mathbf{P}$ is not bounded. Its quite important to note that we are using faulty English, not bounded does not necessarily mean unbounded. We will again use the goated supremum norm to do this.

$$|\mathbf{P}| = \sup_{f \in C^1} \frac{||\mathbf{P}f||_\infty}{||f||\infty} \geq \sup_{f \in \{f_n\}_{n\in\mathbb{N}}} \frac{||\mathbf{P}f||_\infty}{||f||\infty} \text{ S.T. } \{f_n\}_{n\in\mathbb{N}} = \{\sin(2\pi n x)\}_{n\in\mathbb{N}} \tag{71}$$

By shrinking the domain of the supremum we can make an inequality that still holds. Considering the $\sin(2\pi n x)$, we know its derivative to be $2\pi n \cos(2\pi n x)$. The norm of the sin is clearly 1 due to it being a bounded function. Then the norm of the derivative is $2\pi n$, letting us make this next statement.

$$\sup_{f \in \{f_n\}_{n\in\mathbb{N}}} \frac{||\mathbf{P}f||_\infty}{||f||\infty} = \sup_{n\in\mathbb{N}} \frac{||\mathbf{P}f_n||_\infty}{||f_n||\infty} = \sup_{n\in\mathbb{N}} 2\pi n = \infty \tag{72}$$

$$\implies |\mathbf{P}| \geq \infty \tag{73}$$

Thus showing the momentum operator is not bounded!!!! Yayyy, eat that Heisenberg. Commute this bitch. Well, I guess this example isn't exactly the same. In quantum mechanics we use a Hilbert space and not a Banach space (But all Hilbert Spaces are Banach??) and we also don't use the infinity norm. Let this be a harbinger of the fact that the true momentum operator is not bounded and neither is the position operator.
Let us look at a particular space and see if it is Banach. This theorem is as follows: The set

$$\mathcal{L}(V,W) := \left\{\mathbf{A} : V \to W \,\middle|\, \mathbf{A} \text{ is linear and bounded}\right\} \tag{74}$$

forms a Banach space if you use the operator norm in tandem with a scalar multiplication and vector addition defined point wise in terms of W. Again here, we must show CANI and ADDU alongside the fact that the operator norm is indeed a norm. These are relatively minor and can be shown as follows: Now for completeness. We can take the same strategy as before. Assume $\mathbf{A}_n$ forms a Cauchy sequence in $\mathcal{L}(V, W)$.

1. Construct a candidate map $\mathbf{A}$ for the limit of the sequence
2. Show that $\mathbf{A}$ is linear and bounded
3. Show that $\mathbf{A}_n \to \mathbf{A}$.

With this lined up lets knock em out one by one.

1. Construct a candidate map $\mathbf{A}$ for the limit of the sequence

   Because $\{\mathbf{A}_n\}$ is Cauchy in $\mathcal{L} \implies \forall f \in V : \mathbf{A}_n f_{n \in \mathcal{N}}$ is Cauchy $\subseteq W$. This is true because of our use of the operator norm!

$$\forall \epsilon > 0 \exists N \in \mathbb{N}\ \forall n, m > N : ||\mathbf{A}_n - \mathbf{A}_m|| < \epsilon \tag{75}$$
$$\implies \forall f \in V : ||\mathbf{A}_n f - \mathbf{A}_m f||_W = ||(\mathbf{A}_n - \mathbf{A}_m) f|| \tag{76}$$
$$\leq ||\mathbf{A}_n - \mathbf{A}_m|| \cdot ||f||_V < \epsilon \cdot ||f||_V \tag{77}$$

   Where the last two inequalities hold because the operator norm uses the supremum. If we blindly pick an f, then it certainly is less than the supremeum of those f. Then because f is bounded we can simply say that the norm of $||f||_V$ is finite. All together showing that it is indeed Cauchy in W. Using this fact and the fact that W is Banach, we may conclude that $\lim_{n\to\infty}(\mathbf{A}_n f)$ exists $\in W$. Then we can define our candidate map as this limit. Namely:

$$\mathbf{A} : V \to W \tag{78}$$
$$f \mapsto \lim_{n\to\infty}(\mathbf{A}_n f) \tag{79}$$

2. Show that $\mathbf{A}$ is linear and bounded

   Linearity is easy. Simply consider the fact that $\mathbf{A}$ is built by taking the limit of the $\mathbf{A}_n$'s. In order for a map to be linear, when acting on the sum of the inputs it must be equal to summing the inputs after having been acted on. Because each $\mathbf{A}_n$ is linear, all that is needed is to ensure that the limit of the sum is the sum of the limits. This is of course true because the $+_W$ is a continuous map. Therefore implying that the limit map is linear. If this doesn't make sense to you, go to min 55 of lecture 2 of Schuller. What is left to show is to show that $\mathbf{A}$ is bounded.

$$
\begin{aligned}
&\forall f \in V && (80)\\
&||\mathbf{A}f||_W = ||\lim_{n\to\infty}(\mathbf{A}_n f)||_W = \lim_{n\to\infty}||\mathbf{A}_n f||_W && (81)
\end{aligned}
$$

Where the last equal sign is only valid if the norm is continuous.

$$\lim_{n\to\infty}||\mathbf{A}_n f||_W \leq \lim_{n\to\infty}||\mathbf{A}_n||||f||_V \tag{82}$$

This inequality is true because of the use of the supremum inside the operator norm. We picked a specific f in the beginning, this means that if we take the supremum of all fs it must be less than or equal. We know that $||f||_V$ is finite because we know that f is contained within V. $\lim_{n\to\infty}||\mathbf{A}_n||$ is bounded due to the sequence of $\mathbf{A}_n$s being Cauchy. It is true that:

$$
\begin{aligned}
&\forall \epsilon > 0 \exists N \in \mathbb{N}\ \forall n, m > N : ||\mathbf{A}_n - \mathbf{A}_m|| < \epsilon && (83)\\
&\implies \forall n > N : ||\mathbf{A}_n - \mathbf{A}_N|| < \epsilon && (84)\\
& && (85)
\end{aligned}
$$

By the triangle identity one can see:

$$
\begin{aligned}
&||\mathbf{A}_n - \mathbf{A}_N|| \geq ||\mathbf{A}_n|| - ||\mathbf{A}_N|| && (86)\\
&\implies ||\mathbf{A}_n|| < \epsilon||\mathbf{A}_N|| && (87)
\end{aligned}
$$

If we then take the limit as n goes to infinity we can see that it is indeed bounded because we may pick any epsilon and $||\mathbf{A}_N||$ is an element of the bounded maps. This allows us to make this inequality

$$\forall f \in V \frac{||\mathbf{A}f||_W}{||f||_V} < \tag{88}$$

Implying that our candidate map belongs in the $\mathcal{L}$ space.

3. Show that $\mathbf{A}_n \to \mathbf{A}$.

   We lastly need to show that our Cauchy sequence exactly converges to our candidate map. Since $\{\mathbf{A}_n\}$ is Cauchy it is true that:

$$\forall \epsilon > 0 \exists N \in \mathbb{N}\ \forall n, m > N : ||\mathbf{A}_n - \mathbf{A}_m|| < \epsilon \tag{89}$$

Again, we can do shenanigans with the supremum. If it holds for the supremum, it holds for a specific f. This implies:

$$\forall\epsilon > 0 \exists N \in \mathbb{N}\ \forall n, m > N : \forall f \in V : \frac{||(\mathbf{A}_n - \mathbf{A}_m)f||_W}{||f||_V} < \epsilon \tag{90}$$

As Schuller notes, it is important to keep the order of the qualifying statements before the inequality. There exists one for all is not the same as for all there exists one. By using the continuity of the norm, we can then take a limit as m goes to infinity. Letting us then say:

$$\forall\epsilon > 0 \exists N \in \mathbb{N}\ \forall n > N : \forall f \in V : \frac{||(\mathbf{A}_n - \mathbf{A})f||_W}{||f||_V} < \epsilon \tag{91}$$

Which holds for all f, implying it certainly holds for the supremum of fs. Therefore showing that our space is Banach QED.

2.3 Bounded Linear Transformation Theorem

In our terminology we might call this the bounded linear map theorem. A transformation is a map and vice versa. The BLT theorem most likely holds due to the sandwich! This theorem contains some important details about the domains of maps. To start, consider a bounded linear map that is densely defined on some normed vector space V.

$$\mathbf{A} : \mathcal{D}_{\mathbf{A}} \to W \tag{92}$$

$$\text{Such that } \mathcal{D}_{\mathbf{A}} \subseteq V \text{ and } \overline{\mathcal{D}_{\mathbf{A}}} = V \tag{93}$$

Where the overline on the domain implies closing the set $\mathcal{D}_{\mathbf{A}}$. The question then remains, does there exists an extension $\hat{\mathbf{A}}$ of the map $\mathbf{A}$ such that is is linear and:

$$\hat{\mathbf{A}} : V \to W \tag{94}$$

$$\forall\alpha \in V : \hat{\mathbf{A}}\alpha = \mathbf{A}\alpha \tag{95}$$

Extension is a technical term. What qualifies as an extension? As seen in the mathematical notation above, the extension simply must be bounded as well as agree with the original map on the original maps domain. This then leads us to the BLT theorem. It says: There is a unique such extension of $\mathbf{A}$ to a bounded $\hat{\mathbf{A}}$. In general we can, and always will, extend densely defined bounded linear maps to the entire space. The proof of the theorem is as follows: For $\mathbf{A} \in \mathcal{L}(V, W)$ and $\mathcal{D}_{\mathbf{A}} \subseteq W$

Let $f \in V$ and consider a sequence $\{f_n\} \subseteq \mathcal{D}_{\mathbf{A}}$ with $\lim_{n\to\infty} f_n = f$ (Where we know this equals sign is true due the fact that the domain of A is dense in V). This implies $\{f_n\}$ is certainly Cauchy on $\mathcal{D}_{\mathbf{A}}$. Because $\mathbf{A}$ is bounded we know that the sequence $\{_n\}$ is also Cauchy on W. Noting the fact that every element can have the map applied to it as it is defined to be in its domain. This is true because W is Banach. Since W is Banach we know that the limit

$$\lim_{n\to\infty} \mathbf{A} f_n \in W \tag{96}$$

Allowing us to define the extension as:

$$\hat{\mathbf{A}} : V \to W \tag{97}$$

$$f \mapsto \hat{\mathbf{A}} f := \lim_{n\to\infty} \mathbf{A} f_n \text{ such that } \lim_{n\to\infty} f_n = f \tag{98}$$

These questions then remain, is it an extension, is it bounded, is it linear, is it well defined, and is it unique?

1. Is it an extension?

   Consider an $\alpha \in \mathcal{D}_{\mathbf{A}}$. If we act on on alpha with $\hat{\mathbf{A}}$ we obtian this expression.

$$\hat{\mathbf{A}}\alpha = \lim_{n\to\infty} \mathbf{A} f_n = \mathbf{A}(\lim_{n\to\infty} f_n) = \mathbf{A}\alpha \tag{99}$$

   The limit can be moved because the map is continuous and then we know that the sequence of f_n must equal alpha as supposed in the definition. Therefore showing that it equals the regular $\mathbf{A}$ on $\alpha \in \mathcal{D}_{\mathbf{A}}$

2. Is it bounded?

$$||\hat{\mathbf{A}}|| = \sup_{f\in V} \frac{||\hat{\mathbf{A}} f||_W}{||f||_V} \tag{100}$$

   It is clear that if f is $\in \mathcal{D}_{\mathbf{A}}$ their norms will be equal because it is an extension. What if f is not is $\in \mathcal{D}_{\mathbf{A}}$?

$$\sup_{f\in V} \frac{|| \lim_{n\to\infty} \mathbf{A} f_n||_W}{||f||_V} = \sup_{f\in V} \lim_{n\to\infty} \frac{||\mathbf{A} f_n||_W}{||f||_V} \tag{101}$$

   By the continuity of the norm we can pull the limit out. We know the bottom norm of the f is finite so we will only deal with the top for now.

$$= sup_{f \in V} \lim_{n \to \infty} ||\mathbf{A} f_n||_W \leq sup_{f \in V} \lim_{n \to \infty} ||\mathbf{A}|| ||f_n||_V = ||\mathbf{A}|| ||f||_V < \infty \tag{102}$$

Where the inequality holds due to the definition of the operator norm on $\mathbf{A}$ and because A is certainly bounded that implies $\hat{\mathbf{A}}$ is also.

3. Is it linear?

$$\hat{\mathbf{A}} f + \hat{\mathbf{A}} g = \lim_{n \to \infty} \mathbf{A} f_n + \lim_{n \to \infty} \mathbf{A} g_n \tag{103}$$

Because + is continuous we can now pull the limit into one. The same argument can be shown for $\mathbf{A}$ because it's linear and bounded implying continuity.

$$= \lim_{n \to \infty} \mathbf{A}(f_n + g_n) \tag{104}$$

We know that $\lim_{n \to \infty} f_n = f$ and $\lim_{n \to \infty} g_n = g$. This means:

$$\forall \epsilon > 0 \;\; \exists N_1 \in \mathbb{N} \forall n > N_1 ||f - f_n|| \leq \epsilon/2 \tag{105}$$
$$\forall \epsilon > 0 \;\; \exists N_2 \in \mathbb{N} \forall n > N_2 ||g - g_n|| \leq \epsilon/2 \tag{106}$$
$$\tag{107}$$

If we choose $N = \max\{N1, N2\}$ then for all $n > N$

$$\forall n > N ||(f+g) - (f_n + g_n)|| = ||f - f_n + g - g_n|| \leq ||f - f_n|| + ||g - g_n|| < \epsilon \tag{108}$$

This implies that they will be the same element.

$$= \lim_{n \to \infty} \mathbf{A}(f_n + g_n) = \hat{\mathbf{A}}(f + g) \tag{109}$$

Showing linearity in vector addition. Similarly we can show scalar multiplication.

4. Is it well defined?

Suppose there are two sequences $\{f_n\}$ and $\{g_n\}$ such that both of their limits are f. This implies:

$$\forall\epsilon > 0 \;\; \exists N_1 \in \mathbb{N} \forall n > N_1 ||f - f_n|| \leq \epsilon \tag{110}$$
$$\tag{111}$$
$$\forall\epsilon > 0 \;\; \exists N_2 \in \mathbb{N} \forall n > N_2 ||f - g_n|| \leq \epsilon \tag{112}$$
$$\tag{113}$$

Then for $n \geq \max\{N_1, N_2\}$:

$$||\mathbf{A}f_n - \mathbf{A}g_n||_W \leq ||A||||f_n - g_n||_V = \leq ||A||||f_n - f + f - g_n||_V \tag{114}$$
$$\leq ||A||||f_n - f||_V + ||f - g_n||_V < \leq ||A||2\epsilon \tag{115}$$

Thus in the limiting case:

$$\lim_{n\to\infty}(\mathbf{A}f_n - \mathbf{A}g_n) = 0 \tag{116}$$

But we know that the minus is a continuous map meaning we can move the limit to each element in the subtraction. Giving:

$$\lim_{n\to\infty} \mathbf{A}f_n = \lim_{n\to\infty} \mathbf{A}g_n \tag{117}$$

Showing that it is indeed well-defined.

5. Is it unique?

   Suppose that there is another map $\hat{\mathbf{T}}$ that is an extension of $\mathbf{A}$. Certainly on the domain of $\mathbf{A}$ they are equal. What about an f not in the domain? Consider:

$$||\hat{\mathbf{T}}f - \mathbf{A}f_n|| = ||\hat{\mathbf{T}}f - \hat{\mathbf{T}}f_n|| \leq ||\hat{\mathbf{T}}||||f - f_n|| \tag{118}$$

   This implies that if we take the limit of both sides, the right will approach zero. Showing:

$$\lim_{n\to\infty}(\hat{\mathbf{T}}f - \mathbf{A}f_n) = 0 \tag{119}$$

   By the continuity of minus and the fact that the left element doesnt depend on n we gain:

$$\hat{\mathbf{T}}f = \lim_{n\to\infty} \mathbf{A}f_n := \hat{\mathbf{A}}f \tag{120}$$

   Hence it is unique. QED.

2.4 The Dual Space

The dual space is important in defining how most physicists view quantum mechanics. We will see later that we can take advantage of the dual space as well as the Riesz lemma to define what a 'Bra' means to a 'Ket'. We can start by considering just a normed vector space V. Then we define its dual space as:

$$(V^*, ||\cdot||_{V^*}) = (\mathcal{L}(V, \mathbb{C}), ||\cdot||_{\mathcal{L}}) \tag{121}$$

The dual space is the set of bounded linear functionals from your vector space to the complex numbers. Where its norm is given by the operator norm. Remember that this space IS a Banach space. That is quite useful. The dual space has more structure than the regular vector space.
Lastly, I'm not gonna write this one out. However, if you use the same definition of convergence as we have been using we are going to now rename this to strong convergence. Whereas weak convergence is essentially the same except they only need to converge strongly inside the the complex numbers for all elements of our dual space.

2.5 Problems to Try

1. Prove or disprove any bounded map is necessarily continuous.
2. Prove or disprove any bounded and linear map is necessarily continuous.
3. Show that strong convergence implies weak convergence.

3 Separable Hilbert Spaces

3.1 The Relationship Between Norms, Inner Products, and Hilbert Spaces

Recall that a Hilbert space is a complex vector space $(\mathcal{H}, +, \cdot)$ equipped with a sesqui-linear inner product which will induce a norm with respect to which $(\mathcal{H}, +\cdot, \langle\cdot,\cdot\rangle)$ is complete. We can observe some properties of the Hilbert space using the norm defined with the inner product.

$$||\cdot||_{\mathcal{H}} \to R; \quad ||\psi|| := \sqrt{\langle\psi, \psi\rangle} \tag{122}$$

By creating the norm as seen above we can assert that $(\mathcal{H}, +\cdot, ||\cdot||_{\mathcal{H}})$ is Banach. Allowing us to use the theorems shown in the last lecture. Thus, we know $(\mathcal{L}(\mathcal{H}, \mathcal{H}), ||\cdot||_{\mathcal{L}})$ (The set of bounded linear maps from H to H and the operator norm) forms a Banach space. We also know that the dual space of a Hilbert space $\mathcal{H}^* := (\mathcal{L}(\mathcal{H}, \mathbb{C}), ||\cdot||_{\mathcal{H}^*})$ will make a Banach space. We will also even see that the dual space is even a Hilbert space, i.e. $\exists\, \langle\cdot,\cdot\rangle : ||\psi||_{\mathcal{H}^*} =$

$\sqrt{\langle\psi,\psi\rangle_{\mathcal{H}^*}}$. This leads us to the theorem: A norm $||\cdot||_V : V \to \mathbb{R}$ is induced by a sesqui-linear inner product if and only if the parallelogram identity holds.

$$||f+g|| + ||f-g|| = 2||f|| + 2||g|| \forall f, g \in V \tag{123}$$

If it does hold, we can construct the inner product using the polarization identity.

$$\langle f, g\rangle := \frac{1}{4}(||f+g|| - ||f-g|| + i||f-ig|| - i||f+ig||) \tag{124}$$

Look at the first three problems to see some details.

3.2 Hamel Basis vs Schauder Basis

The notion of a basis for a naked vector space $(V, +, \cdot)$ is knonw from linear algebra.

A subset $B \subseteq V$ is called a (Hamel) basis if

1. Any finite subset $\{e_1, e_2, ...e_n\} \subseteq B$ is linearly independent:

$$\sum_{i=1}^{N} \lambda^i e_i = 0 \implies \lambda^1 = \lambda^2 = ... = \lambda^n = 0 \tag{125}$$

2. For any $v \in V$ a finite subset $\{e_1, e_2, ...e_n\} \subseteq B$ and complex numbers $\{v_1, v_2, ...v_n\}$ s.t.

$$\sum_{i=1}^{N} v^i e_i = v \tag{126}$$

This is not the basis that QM uses. Hilbert spaces have more structure then a naked vector space. We can use their structure to define a refined notion of a basis.

Let $(W, ||\cdot||_W)$ be a Banach space. A Schauder basis of W is a sequence $\{e_n\}_{n\in\mathbb{N}}$ in W such that, for any $f \in W$, there exists a unique sequence $\{\psi^i\}_{n\in\mathbb{N}}$ such that:

$$f = \lim_{n\to\infty} \sum_{i=0}^{n} \psi^i e_i = \sum_{i=1}^{\infty} \psi^i e_i \tag{127}$$

Interestingly enough Schauder basis require a notion of convergence under the topological structure of the vector space and they need non exist for a given vector space (unlike Hamel). Generally, these sets S can be countable and non-countable, this is important in the definition of separable Hilbert spaces which are important to Quantum Mechanics.
I am going to also note the following property of the Schauder Basis. First a definition is in order.

$$\text{span}(U) = \{\text{All } \underline{\text{finite}} \text{ linear combinations of elements } \in U\} \tag{128}$$

If we take a Shauder basis S of $\mathcal{H}$, then the span(S) is dense in $\mathcal{H}$.

3.3 Separable Hilbert Spaces

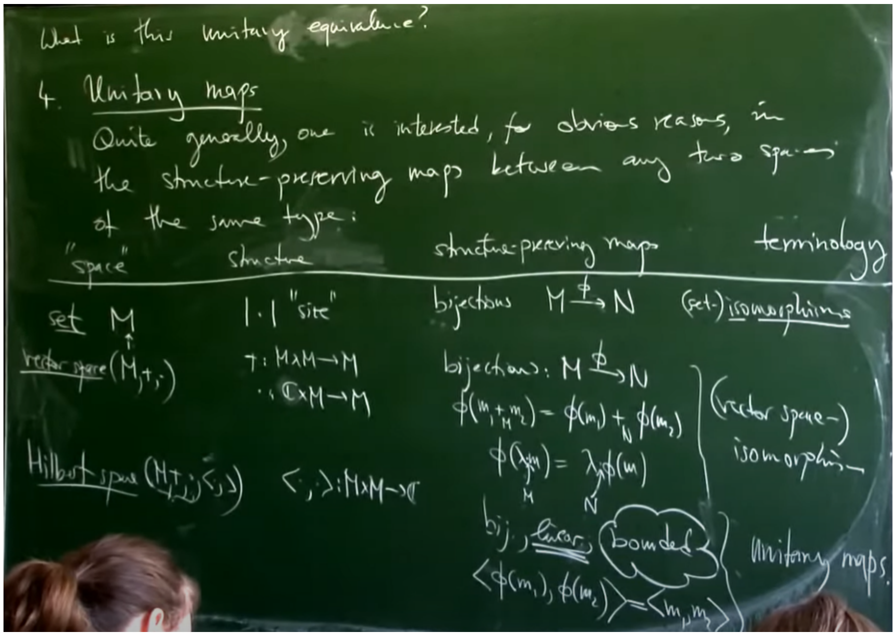
In quantum mechanics proper, we always look at separable Hilbert spaces. What exactly does it mean to be separable? A Hilbert space is separable if it admits a Shauder basis that has the following traits:

1. The Shauder basis is countable
2. The Shauder basis is orthonormal, in that $\langle e_i, e_j \rangle = \delta_{i,j}$

Where $\delta_{i,j}$ is understood as the Kronecker delta symbol. Given the above, how many separable Hilbert spaces are there? (dim $=\infty$). We can attempt to answer this by proving or disproving the following claim: Any infinite dimensional Hilbert space is unitarily isomorphic to the space separable Hilbert space $\ell^2(\mathbb{N})$ of square summable sequences in $\mathbb{C}$.

$$\ell^2(\mathbb{N}) := \{a : \mathbb{N} \to \mathbb{C} \Big| \sum_{i=1}^{\infty} |a_i|^2 < \infty\} \tag{129}$$

We can verify that this forms a vector space with the sesqui-linear inner product. The addition and s-multiplication are quite obvious. If we define the inner product as $\langle a, b \rangle := \sum_{i=1}^{\infty} \overline{a_i} b_i$ we will arrive at a complete Hilbert space.With all the definitions out of the way, the question remains, what is all of this unitary BS. Rather than write this part out I will be pasting a screenie of the chart Schuller made.



In general, when you are looking at a space you look at through the glasses of its structure. When you apply a map to a space, you are interested in how it changes said structure. Of course, to physics the most interesting maps are those that preserve the structure. Think of a change from Cartesian to polar co-ordinates. This chart above outlines the details involved in these maps. For Hilbert spaces, it is called Unitary. To put it clearly: Definition: Let $\mathcal{H}$ and $\mathcal{G}$ be Hilbert spaces. Then any bijection:

$$u \in \mathcal{L}(\mathcal{H}, \mathcal{G}) \tag{130}$$

Such that the inner product is preserved, i.e.

$$\forall \varphi, \psi \in \mathcal{H}: \ \langle u\psi, u\varphi \rangle_{\mathcal{G}} = \langle \psi, \varphi \rangle_{\mathcal{H}} \tag{131}$$

Is called a Unitary map. It is clear, if such a map exists for $\mathcal{H}$ and $\mathcal{G}$ then the two spaces can not be structurally distinguished by only using $(+, \cdot, \langle \cdot, \cdot \rangle)$. Looking through the glasses of Hilbert space theory, they are not distinguishable. If such a map exists, $\mathcal{H}$ and $\mathcal{G}$ are called unitarily equivalent. By the polarization formula, it suffices to check if the map $u \in \mathcal{L}(\mathcal{H}, \mathcal{G})$ preserves the norms, i.e.

$$\forall \psi \in \mathcal{H}: \ ||\psi||_{\mathcal{H}} = ||u\psi||_{\mathcal{G}} \tag{132}$$

If we hope to prove anything, we can only prove it up to unitary equivalence. With this out of the way, we can shown the equivalence: Every separable Hilbert space is unitary equivalent to the Hilbert $(\mathcal{H}, +, \cdot, \langle\cdot,\cdot\rangle)$ space $(\ell^2(\mathbb{N}), \langle\cdot,\cdot\rangle_{\ell^2})$. This is clearly the strongest type of classification. There is only ONE class. Not 7, not a family, not one for each integer, JUST ONE. Proof: Need to show a bounded linear map $u : \mathcal{H} \to \ell^2(\mathbb{N})$ such that $||u\psi||_\ell^2 = ||\psi||_{\mathcal{H}}$. We can construct it directly: Choose an orthonormal Schauder basis S.

$$u : \mathcal{H} \to \ell^2(\mathbb{N}) \tag{133}$$

$$\psi \mapsto \{\langle e_n, \psi\rangle\}_{n\in\mathbb{N}} \tag{134}$$

Clearly the sequence that results after mapping with u is in the complex numbers. If it it square summable then we know that it will be an element of our space ℓ^2, i.e.

$$\sum_{i=1}^{\infty} |\,\langle e_i, \psi\rangle\,|^2 < \infty \tag{135}$$

Because psi can be written in terms of the basis we know:

$$||\psi||_{\mathcal{H}}^2 = ||\sum_{n=1}^{\infty} \langle e_n, \psi\rangle\, e_n||_{\mathcal{H}}^2 \tag{136}$$

By the continuity of the norm we can pull out the implicit limit in the sum

$$= \lim_{N\to\infty} ||\sum_{n=1}^{N} \langle e_n, \psi\rangle\, e_n||_{\mathcal{H}}^2 \tag{137}$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} ||\,\langle e_n, \psi\rangle\, e_n||_{\mathcal{H}}^2 \tag{138}$$

Where the last step is using the Pythagorean theorem given that $\langle e_i, e_j\rangle = \delta_{i,j}$.

$$= \lim_{N\to\infty} \sum_{n=1}^{N} |\,\langle e_n, \psi\rangle\,|^2 ||e_n||_{\mathcal{H}}^2 \qquad = \sum_{n=1}^{\infty} |\,\langle e_n, \psi\rangle\,|^2 = ||u\psi||_{\ell^2}^2 \tag{139}$$

Thus showing that the map will preserve the norm as well as the fact that the resulting sequence is square summable $\implies u\psi \in \ell^2(\mathbb{N})$. We can define another map u^{-1} :

$$u^{-1} : \ell^2(\mathbb{N}) \to \mathcal{H} \tag{140}$$

$$\{a_n\}_{n\in\mathbb{N}} \mapsto \sum_{n=1}^{\infty} a_n e_n \tag{141}$$

Clearly:

$$u^{-1}(u\psi) = \sum_{n=1}^{\infty} \langle e_n, \psi \rangle \psi = \psi \tag{142}$$

$$\implies u^{-1} \circ u = \mathbf{I} = u \circ u^{-1} \tag{143}$$

Because u is unitary, if we look at the operator norm of u we can see that it must equal 1.

$$||u||_{\mathcal{L}(\mathcal{H},\ell^2)} = \sup_{f\in\mathcal{H}} \frac{||uf||_{\ell^2}}{||f||_{\mathcal{H}}} = \sup_{f\in\mathcal{H}} \frac{||f||_{\mathcal{H}}}{||f||_{\mathcal{H}}} = 1 \tag{144}$$

Where the swapping of the numerator is due to the preservation of the norm under a unitary map. This shows that we have only 1 separable Hilbert Space. QED.

3.4 Problems to Try

1. Prove that that if the parallelogram identity holds, the sesqui-linear inner-product induces the norm.
2. Prove $\mathcal{H}^*$ is a Hilbert space.
3. Prove the Cauchy-Schwarz inequality, namely $| \langle \psi, \phi \rangle | \leq ||\psi||||\phi||$